Lecture note for Magnetism

W. Zhu[1]

[1] *Westlake Institute of Advanced Study,*

*Westlake University, Hangzhou, P. R. China**

Contents

[7]Reporting of typos, inaccuracies and errors to zhuwei@westlake.edu.cn would be greatly appreciated.

NON-LINEAR SIGMA MODEL

For continuous spins, the lowest energy excitations are long-wavelength Goldstone modes (as shown in the Laudau theory). We consider unit n-component spins on the sites of a lattice, i.e.

$$\mathbf{s}_i = (s_1, s_2, ..., s_n), |\mathbf{s}_i|^2 = s_1^2 + s_2^2 + ... + s_n^2 = 1 \tag{1}$$

The nearest-neighbor Hamiltonian can be written as

$$-\beta H = K \sum_{\langle i,j \rangle} \mathbf{s}_i \cdot \mathbf{s}_j = K \sum_{\langle i,j \rangle} (1 - \frac{(\mathbf{s}_i - \mathbf{s}_j)^2}{2}) \approx -\beta E_0 - \frac{K}{2} \int d^d x (\nabla s(\mathbf{x}))^2. \tag{2}$$

Therefore the partition function is

$$Z = \int D[\mathbf{s}(x)] \delta(|\mathbf{s}(x)|^2 - 1) e^{-\frac{K}{2} \int d^d x (\nabla \mathbf{s}(\mathbf{x}))^2}. \tag{3}$$

The non-linear sigma model is closed to the O(N) vector model.

$$S = \int d^d x [|\nabla \Psi(x)|^2 + r\Psi^2 + u(\Psi^2 - \mathbf{s}_0)^2] \tag{4}$$

For the critical point, $|\mathbf{s}_0|$ vanishes so it recovers the critical $O(N)$ model. In the study of ordered phase, $|s_0|$ is non-zero, we can take a mean-field approximation ($\Psi \sim \mathbf{s}_0$) to get $S = \int d^d x |\nabla \mathbf{s}_0(x)|^2$.

For a possible magnetic order, say $\mathbf{s}(\mathbf{x}) = (1, 0, 0, ...)$, there are $n-1$ Goldstone modes describing the transverse fluctuations. To examine these fluctuations close to zero temperature, we set

$$\mathbf{s}(\mathbf{x}) = (\sigma(x), \pi_1(x), \pi_2(x), ...) \equiv (\sigma(x), \vec{\pi}(x)) \tag{5}$$

where $\vec{\pi}$ is an $n-1$ component vector describing transverse fluctuations. For each degree of freedom,

$$\begin{aligned} \int D[\mathbf{s}] \delta(|s|^2 - 1) &= \int d\vec{\pi} d\sigma \delta(\pi^2 + \sigma^2 - 1) \\ &= \int d\vec{\pi} d\sigma \delta((\sigma - \sqrt{1-\pi^2})(\sigma + \sqrt{1-\pi^2})) = \int d\vec{\pi} \frac{1}{2\sqrt{1-\pi^2}} \end{aligned} \tag{6}$$

Using this result, the partition function can be written as

$$\begin{aligned} Z &= \int D[\mathbf{s}(x)]\delta(|\mathbf{s}(x)|^2-1)e^{-\frac{K}{2}\int d^dx(\nabla \mathbf{s}(\mathbf{x}))^2} \\ &= \int d\vec{\pi}\frac{1}{2\sqrt{1-\pi^2}}e^{-\frac{K}{2}\int d^dx(\nabla\pi(\mathbf{x}))^2+(\nabla\sqrt{1-\pi^2(\mathbf{x})})^2} \\ &= \int d\vec{\pi}\exp[-\int d^dx[\frac{K}{2}(\nabla\pi(\mathbf{x}))^2+\frac{K}{2}(\nabla\sqrt{1-\pi^2(\mathbf{x})})^2+\frac{\rho}{2}\ln(1-\pi^2)]] \end{aligned} \tag{7}$$

where $\rho = N/V$ is density of lattice sites. Here we see, while the original Hamiltonian is quite simple, the effective Hamiltonian describing Goldstone modes $\vec{\pi}(x)$ is rather complicated.

We can expand the nonlinear terms in powers of $\vec{\pi}(x)$, resulting in a series

$$\beta H[\vec{\pi}(x)] \approx \beta H_0 + U_1 + U_2 + ... \tag{8}$$

$$H_0 = \frac{K}{2}\int d^dx(\nabla\pi(\mathbf{x}))^2 \tag{9}$$

$$U_1 = \int d^dx[\frac{K}{2}(\pi(\mathbf{x})\cdot\nabla\pi(\mathbf{x}))^2 - \frac{\rho}{2}\pi^2(\mathbf{x})] \tag{10}$$

Here H_0 is independent Goldstone modes, and U_1 is the first order perturbation when the terms in the series due to $\ln(1-x)\approx -x, \nabla\sqrt{1-x^2} = \frac{x\nabla x}{\sqrt{1-x^2}} \approx x\nabla x$.

In the language of Fourier modes,

$$\beta H_0 = \frac{K}{2}\int\frac{d^dq}{(2\pi)^d}q^2|\vec{\pi}(\mathbf{q})|^2 \tag{11}$$

$$\begin{aligned} U_1 = &-\frac{K}{2}\int\frac{d^dq_1}{(2\pi)^d}\frac{d^dq_2}{(2\pi)^d}\frac{d^dq_3}{(2\pi)^d}\frac{d^dq_4}{(2\pi)^d}\pi_\alpha(\mathbf{q}_1)\pi_\alpha(\mathbf{q}_2)\pi_\beta(\mathbf{q}_3)\pi_\beta(\mathbf{q}_4)(\mathbf{q}_1\cdot\mathbf{q}_3)\delta(\mathbf{q}_1+\mathbf{q}_2+\mathbf{q}_3+\mathbf{q}_4) \\ &-\frac{\rho}{2}\int\frac{d^dq}{(2\pi)^d}|\vec{\pi}(\mathbf{q})|^2 \end{aligned} \tag{12}$$

For non-interacting part, the correlation function of the Goldstone modes are

$$\langle\pi_\alpha(q)\pi_\beta(q')\rangle_0 = \frac{\delta_{\alpha\beta}(2\pi)^d\delta(\mathbf{q}+\mathbf{q}')}{Kq^2} \tag{13}$$

$$\begin{aligned} &\to \langle\pi(x=0)^2\rangle = \int\frac{d^dq}{(2\pi)^d}\langle|\pi_\alpha(q)|^2\rangle_0 \\ &\frac{n-1}{K}\int_{1/L}^{\Lambda=1/a}\int\frac{d^dq}{(2\pi)^d}\frac{1}{q^2} = \frac{n-1}{K}\frac{S^d(a^{2-d}-L^{2-d})}{d-2} \end{aligned} \tag{14}$$

For $d > 2$, the fluctuations are proportional to T, but for $d < 2$, the fluctuations diverge since $L \to \infty$. This is a direct sequence of Mermin-Wagner theorem on the absence of long-range order in $d \leq 2$.

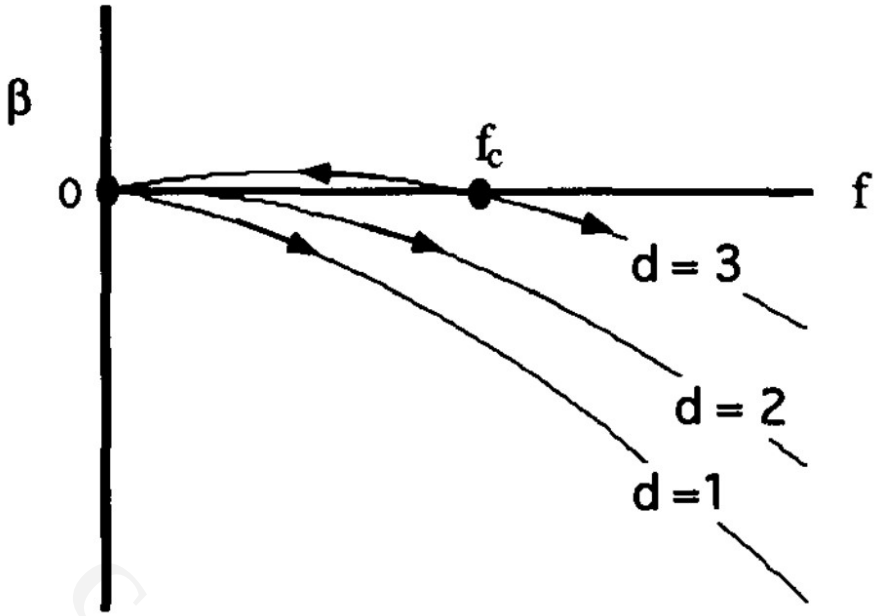


FIG. 1: RG diagram of the non-linear sigma model for space-time dimension d and n-component.

An RG expansion in powers of T provides a systematic way to explore critical behavior close to two dimensions. By skipping the RG details, one can obtain the RG equation for the temperature as

$$\frac{dK}{d\ell} = (d-2)K - (n-2)S^d\Lambda^{d-2} \leftrightarrow \frac{dT}{d\ell} = -\frac{dK}{K^2 d\ell} = -(d-2)T + (n-2)S^d\Lambda^{d-2}T^2 \quad (15)$$

The RG equation changes behavior drastically at $d = 2$. For $d < 2$, the (the leading order) temperature flow is away from zero, indicating that the ordered phase is unstable and there is no broken symmetry. For $d > 2$, small T flows back to zero, implying that the ordered phase is stable. At $d = 2$, the flow is controlled by the second term, which changes sign at $n = 2$. For $n > 2$ the flow is towards high temperatures, so that Heisenberg and higher spin models are disordered. The situation of $n = 2$ is special, and it is marginal in the RG calculations. This special case will be discussed in more detail in the following sections.

Renormalization group

Next we use the Wilson's renormalization group to calculate the flow of coupling parameter. We divide the mode $\pi_\alpha(q)$ into two parts: slow mode $0 < q < \Lambda/b$ and fast

mode $\Lambda/b < q < \Lambda$. The coarse-grained Hamiltonian Eq. 11 is given by

$$\beta H[\pi^<] = \beta H_0[\pi^<] + \langle U_1[\pi^< + \pi^>]\rangle_0 \tag{16}$$

where $\langle U_1[\pi^< + \pi^>]\rangle_0$ means averaging over fast mode $\pi^>$.

We write the first-order expansion

$$\begin{aligned}
\langle U_1[\pi^< + \pi^>]\rangle_0 &\approx -\frac{K}{2}\int \frac{d^dq_1}{(2\pi)^d}\frac{d^dq_2}{(2\pi)^d}\frac{d^dq_3}{(2\pi)^d}\frac{d^dq_4}{(2\pi)^d}(\mathbf{q}_1\cdot\mathbf{q}_3)\delta(\mathbf{q}_1+\mathbf{q}_2+\mathbf{q}_3+\mathbf{q}_4) \\
&\quad (\pi^<_\alpha(\mathbf{q}_1)+\pi^>_\alpha(\mathbf{q}_1))(\pi^<_\alpha(\mathbf{q}_2)+\pi^>_\alpha(\mathbf{q}_2))(\pi^<_\beta(\mathbf{q}_3)+\pi^>_\beta(\mathbf{q}_3))(\pi^<_\beta(\mathbf{q}_4)+\pi^>_\beta(\mathbf{q}_4)) \\
&\approx -\frac{K}{2}\int \frac{d^dq_1}{(2\pi)^d}\frac{d^dq_2}{(2\pi)^d}\frac{d^dq_3}{(2\pi)^d}\frac{d^dq_4}{(2\pi)^d}(\mathbf{q}_1\cdot\mathbf{q}_3)\delta(\mathbf{q}_1+\mathbf{q}_2+\mathbf{q}_3+\mathbf{q}_4) \\
&\quad [\pi^<_\alpha(\mathbf{q}_1)\pi^<_\alpha(\mathbf{q}_2)\pi^<_\beta(\mathbf{q}_3)\pi^<_\beta(\mathbf{q}_4) \\
&\quad + 2\langle\pi^>_\alpha(\mathbf{q}_1)\pi^>_\alpha(\mathbf{q}_2)\rangle\pi^<_\beta(\mathbf{q}_3)\pi^<_\beta(\mathbf{q}_4) + 2\langle\pi^>_\alpha(\mathbf{q}_1)\pi^>_\beta(\mathbf{q}_4)\rangle\pi^<_\alpha(\mathbf{q}_2)\pi^<_\beta(\mathbf{q}_3) \\
&\quad + \langle\pi^>_\alpha(\mathbf{q}_1)\pi^>_\beta(\mathbf{q}_3)\rangle\pi^<_\alpha(\mathbf{q}_2)\pi^<_\beta(\mathbf{q}_4) + \langle\pi^>_\alpha(\mathbf{q}_2)\pi^>_\beta(\mathbf{q}_4)\rangle\pi^<_\alpha(\mathbf{q}_1)\pi^<_\beta(\mathbf{q}_3)] \\
&= (a)+(b)+(c)+(d)
\end{aligned} \tag{17}$$

Here we have neglected the terms with odd number of $\pi^>$, which should vanish by symmetry. Next we consider the calculation term by term.

The first term, $(a) \sim \pi^<_\alpha(\mathbf{q}_1)\pi^<_\alpha(\mathbf{q}_2)\pi^<_\beta(\mathbf{q}_3)\pi^<_\beta(\mathbf{q}_4)$, is just the quartic term for slow modes. The second term should be zero, because the integral over the shell momentum $\mathbf{q}_1$ is odd:

$$\begin{aligned}
(b) &= -\frac{K}{2}\int \frac{d^dq_1}{(2\pi)^d}\frac{d^dq_2}{(2\pi)^d}\frac{d^dq_3}{(2\pi)^d}\frac{d^dq_4}{(2\pi)^d}(\mathbf{q}_1\cdot\mathbf{q}_3)\delta(\mathbf{q}_1+\mathbf{q}_2+\mathbf{q}_3+\mathbf{q}_4)\langle\pi^>_\alpha(\mathbf{q}_1)\pi^>_\alpha(\mathbf{q}_2)\rangle\pi^<_\beta(\mathbf{q}_3)\pi^<_\beta(\mathbf{q}_4) \\
&= -\frac{K}{2}\int \frac{d^dq_1}{(2\pi)^d}\frac{d^dq_2}{(2\pi)^d}\frac{d^dq_3}{(2\pi)^d}\frac{d^dq_4}{(2\pi)^d}(\mathbf{q}_1\cdot\mathbf{q}_3)\delta(\mathbf{q}_1+\mathbf{q}_2+\mathbf{q}_3+\mathbf{q}_4)\frac{\delta_{\alpha\beta}(2\pi)^d\delta(\mathbf{q}_1+\mathbf{q}_2)}{Kq_1^2}\pi^<_\beta(\mathbf{q}_3)\pi^<_\beta(\mathbf{q}_4) \\
&= -\frac{K}{2}\int \frac{d^dq_1}{(2\pi)^d}\frac{d^dq_3}{(2\pi)^d}(\mathbf{q}_1\cdot\mathbf{q}_3)\frac{\delta_{\alpha\beta}(2\pi)^d}{Kq_1^2}\pi^<_\beta(\mathbf{q}_3)\pi^<_\beta(-\mathbf{q}_3) = 0
\end{aligned} \tag{18}$$

$$\begin{aligned}
(c) &= -\frac{K}{2}\int \frac{d^dq_1}{(2\pi)^d}\frac{d^dq_2}{(2\pi)^d}\frac{d^dq_3}{(2\pi)^d}\frac{d^dq_4}{(2\pi)^d}(\mathbf{q}_1\cdot\mathbf{q}_3)\delta(\mathbf{q}_1+\mathbf{q}_2+\mathbf{q}_3+\mathbf{q}_4)\langle\pi^>_\alpha(\mathbf{q}_1)\pi^>_\beta(\mathbf{q}_3)\rangle\pi^<_\alpha(\mathbf{q}_2)\pi^<_\beta(\mathbf{q}_4) \\
&= -\frac{K}{2}\int_{\Lambda/b}^{\Lambda}\frac{d^d\mathbf{q}_1}{(2\pi)^d}\frac{(2\pi)^d\delta(\mathbf{q}_1+\mathbf{q}_3)(\mathbf{q}_1\cdot\mathbf{q}_3)}{Kq_1^2}\int_0^{\Lambda/b}\frac{d^dq_2}{(2\pi)^d}|\pi^<(\mathbf{q}_2)|^2 \\
&= \frac{\rho}{2}(1-b^{-d})\int_0^{\Lambda/b}\frac{d^dq_2}{(2\pi)^d}|\pi^<(\mathbf{q}_2)|^2
\end{aligned} \tag{19}$$

where $\rho = \int_0^\Lambda \frac{d^d q}{(2\pi)^d}$.

$$
\begin{aligned}
(d) &= -\frac{K}{2}\int \frac{d^d q_1}{(2\pi)^d}\frac{d^d q_2}{(2\pi)^d}\frac{d^d q_3}{(2\pi)^d}\frac{d^d q_4}{(2\pi)^d}(\mathbf{q}_1\cdot\mathbf{q}_3)\delta(\mathbf{q}_1+\mathbf{q}_2+\mathbf{q}_3+\mathbf{q}_4)\langle\pi_\alpha^>(\mathbf{q}_2)\pi_\beta^>(\mathbf{q}_4)\rangle\pi_\alpha^<(\mathbf{q}_1)\pi_\beta^<(\mathbf{q}_3) \\
&= -\frac{K}{2}\int_{\Lambda/b}^{\Lambda}\frac{d^d q_2}{(2\pi)^d}\frac{d^d q_4}{(2\pi)^d}\frac{(2\pi)^d\delta(\mathbf{q}_2+\mathbf{q}_4)}{Kq_2^2}\int_0^{\Lambda/b}\frac{d^d q_1}{(2\pi)^d}(-q_1^2)|\pi^<(\mathbf{q}_1)|^2 \\
&= \frac{1}{2}I_d(b)\int_0^{\Lambda/b}\frac{d^d q_1}{(2\pi)^d}q_1^2|\pi^<(\mathbf{q}_1)|^2
\end{aligned}
\tag{20}
$$

where $I_d(b) = \int_{\Lambda/b}^{\Lambda}\frac{d^d q}{(2\pi)^d}\frac{1}{q^2} = \frac{S_d\Lambda^{d-2}(1-b^{2-d})}{d-2}$.

To sum up, we have

$$
\begin{aligned}
\beta H[\pi^<] = &\frac{K}{2}(1+\frac{I_d(b)}{K})\int\frac{d^d q}{(2\pi)^d}q^2|\vec{\pi}(\mathbf{q})|^2 \\
&-\frac{K}{2}\int\frac{d^d q_1}{(2\pi)^d}\frac{d^d q_2}{(2\pi)^d}\frac{d^d q_3}{(2\pi)^d}\frac{d^d q_4}{(2\pi)^d}\pi_\alpha^<(\mathbf{q}_1)\pi_\alpha^<(\mathbf{q}_2)\pi_\beta^<(\mathbf{q}_3)\pi_\beta^<(\mathbf{q}_4)(\mathbf{q}_1\cdot\mathbf{q}_3)\delta(\mathbf{q}_1+\mathbf{q}_2+\mathbf{q}_3+\mathbf{q}_4) \\
&-\frac{\rho}{2}(1-(1-b^{-d}))\int\frac{d^d q}{(2\pi)^d}|\vec{\pi}(\mathbf{q})|^2
\end{aligned}
\tag{21}
$$

Rescaling $x' = x/b$ and field $\pi'(x) = \pi^<(x)/\zeta$, we obtain the renormalized hamiltonian in real space as

$$
\begin{aligned}
-\beta H = &-\frac{K(1+\frac{I_d(b)}{K})b^{d-2}\zeta^2}{2}\int d^d x'(\nabla\pi'(\mathbf{x}'))^2 \\
&-\frac{Kb^{d-2}\zeta^4}{2}\int d^d x'(\pi'(\mathbf{x}')\nabla\pi'(\mathbf{x}'))^{\mathbf{2}} + \frac{\rho\boldsymbol{\zeta}^{\mathbf{2}}}{\mathbf{2}}\int\mathbf{d}^{\mathbf{d}}\mathbf{x}'(\pi(\mathbf{x}'))^{\mathbf{2}}
\end{aligned}
\tag{22}
$$

The field $\pi(x)$ in non-linear sigma model is constrained by many conditions. ζ can be fixed by the conditions below:

$$
\begin{aligned}
\langle\mathbf{s}\rangle_0 &= \langle(\pi_1^< + \pi_1^>, ..., \sqrt{1-(\pi^<+\pi^>)^2})\rangle_0 \\
&= (\pi_1^<, ..., 1-\frac{(\pi^<)^2}{2}-\frac{\langle(\pi^>)^2\rangle_0}{2}+...) \\
&= (1-\frac{\langle(\pi^>)^2\rangle_0}{2}+...)(\pi_1^<, ..., \sqrt{1-(\pi^<)^2})
\end{aligned}
\tag{23}
$$

The rescaled length is determined to be

$$
\zeta = 1-\frac{\langle(\pi^>)^2\rangle_0}{2} = 1-\frac{(n-1)}{2}\frac{I_d(b)}{K} \tag{24}
$$

with a factor $n-1$ from number of transverse modes.

With this length, we have the rescaled paramter as

$$\begin{aligned} K' &= b^{d-2}\zeta^2 K(1+\frac{I_d(b)}{K}) \\ &= b^{d-2}[1-\frac{(n-1)}{2}\frac{I_d(b)}{K}]^2 K(1+\frac{I_d(b)}{K}) \\ &\approx b^{d-2}K[1-\frac{n-2}{K}I_d(b)] \\ \Rightarrow K+dK &= (1+d\ell)^{d-2}K[1-\frac{n-2}{K}S_d\Lambda^{d-2}d\ell] \end{aligned} \tag{25}$$

In the last line we used $b = 1 + d\ell$. The differential recursion relation corresponding to

$$\frac{dK}{d\ell} = (d-2)K - (n-2)S^d\Lambda^{d-2} \tag{26}$$

$$\leftrightarrow \frac{dT}{d\ell} = -\frac{dK}{K^2 d\ell} = -(d-2)T + (n-2)S^d\Lambda^{d-2}T^2 \tag{27}$$

Based on the RG equation, we can determine the fixed point value

$$T_c = \frac{d-2}{(n-2)S^d\Lambda^{d-2}} + O((d-2)^2). \tag{28}$$

($d = 2, n = 2$ is a special case and one should consider it separately.)

Here we can calculate two eigenvalues. 1) Around this fixed point, the RG equation is linearized as:

$$\frac{d\delta T}{d\ell}|_{T_c} = [-(d-2) + 2(n-2)S^d\Lambda^{d-2}T_c]\delta T \approx [-(d-2)+2(d-2)]\delta T = (d-2)\delta T \tag{29}$$

The eigenvalue of T is $\epsilon = d - 2 \equiv y_t$.

2) The magnetic eigenvalue can be obtained by adding a term $-\mathbf{h}\cdot\int d^dx\mathbf{s}(x)$. Under the RG transformation, we get that

$$h' = b^d\zeta h \equiv b^{y_h}h \tag{30}$$

$$\rightarrow (1+y_h d\ell) = (1+d\ell)^d(1-\frac{(n-1)}{2}\frac{I_d(b)}{K}) \approx (1+d\ell)^d(1-\frac{(n-1)}{2K}S^d\Lambda^{d-2}d\ell) \tag{31}$$

$$y_h = d - T_c\frac{(n-1)}{2}S^d\Lambda^{d-2} = d-(d-2)\frac{n-1}{2(n-2)} = 1+\epsilon\frac{n-3}{2(n-2)} \tag{32}$$

Here we could calculate the critical exponents based on the RG. 1) For correlation length ξ, it becomes divergent close to the critical point. We assume it depends on T

only. Under the one-step RG transformation,

$$b\xi(T') = \xi(T) \tag{33}$$

because length scale increases by a factor of b so ξ reduces by a factor of b. Furthermore, we assume the correlation length takes the form of

$$\xi(T) \sim (T - T_c)^{-\nu} \tag{34}$$

Thus,

$$b((T - T_c)b^{y_t})^{-\nu}) = (T - T_c)^{-\nu} \Rightarrow bb^{-\nu y_t} = 1 \Rightarrow \nu = \frac{1}{y_t} \tag{35}$$

where y_t is the eigenvalue of RG equation by linearization around the fixed point T_c.

2) For the anomalous exponent, we know it is related to the scaling dimension of order parameter. On the one hand,

$$\langle \mathbf{s}(x)\mathbf{s}(0)\rangle = \frac{1}{x^{2\Delta_s}} \tag{36}$$

On the other hand, in the usual definition it is like

$$\langle \mathbf{s}(x)\mathbf{s}(0)\rangle = \frac{1}{x^{d-2+\eta}} \tag{37}$$

Combining these two conditions gives

$$\Delta_s = \frac{d-2+\eta}{2} \tag{38}$$

The so-called 'anomalous' means that, for free theory (Gaussian), $\Delta_s = \frac{d-2}{2}$ so $\eta = 0$. η describes the derivation away from the value of free theory. Based on this, we use Eq. 24

$$\zeta \approx 1 - \frac{(n-1)}{2}\frac{I_d(b)}{K_c} = 1 - \frac{n-1}{2(n-2)}(d-2)d\ell \approx b^{-\Delta_s}, \Delta_s = \frac{n-1}{2(n-2)}(d-2) \tag{39}$$

$$\Rightarrow \eta = 2\Delta_s - (d-2) = \frac{n-1}{n-2}(d-2) - (d-2) = \frac{d-2}{n-2} \tag{40}$$

Mermin-Wegner theorem

Through the study of non-linear sigma model, we can deduce the Mermin-Wagner theorem below: *In one and two dimensions, continuous symmetries cannot be spontaneously broken at finite temperature in systems with sufficiently short-range interactions.* (Please be aware the "continuous" condition here.)

The theorem says that if we try to break the symmetry by imposing a field and then letting the field go to zero, the symmetry remains unbroken in the sense that the average manetization is zero.

1D QUANTUM MAGNETISM

Lieb-Schultz-Mattis theorem

The general LSM theorem: For a lattice model with half-odd-spin in each unit cell, the ground state is either gapless or gapped with degeneracy. So the gapped paramagnet without degeneracy is impossible.

This theorem is very helpful in the study of spin liquid. That is, for spin-1/2 lattice like triangular and kagome, the ground state could be:

- Gapless: 1) magnetic order with spin rotation symmetry spontaneously broken 2) gapless spin liquid.
- Gapped: 1) Valence-bond solid; 2) Gapped spin liquid.

Next we try to elucidate this theorem in one-dimensional spin-1/2 anti-ferromagnetic chain:

$$H = \sum_{\langle ij \rangle} S_i^z S_j^z + \frac{1}{2}(S_i^+ S_j^- + h.c.) \tag{41}$$

Next we go to prove the LSM theorem in spin-1/2 system. Suppose the ground state is $H|\Psi_0\rangle = E_0|\Psi_0\rangle$, where $|\Psi_0\rangle$ is very complicated but we don't need to know it exactly.

We define two operators first. The first operator is translation operator T. Translation operator T commutes with the Hamiltonian,

$$TS_jT^{-1} = S_{j+1}, [T, H] = 0. \tag{42}$$

We assume the eigenvalue of translational operation is

$$T\Psi_0 = e^{ig}\Psi_0 \tag{43}$$

The second operator is U_k for $k = 2\pi/L$:

$$U_k \equiv \exp(ik \sum_n n S_n^z). \tag{44}$$

Here operator U is very important. It has a non-trivial commutation relation with T:

$$TU_kT^{-1} = \exp(ik \sum_n n S_{n+1}^z) = U_k \exp(ikN S_1^z) \exp(-ik \sum_n S_n^z) = -U_k \exp(-ik \sum_n S_n^z) \tag{45}$$

where we used the condition $e^{i2\pi S_1^z} = e^{i\pi\sigma_z} = I\cos\pi + i\sin\pi\sigma_z = -I$. Please note that, this condition can be generalized to

$$e^{i2\pi S^z} = \begin{cases} -1, S = \frac{1}{2}, \frac{3}{2}, ... \\ 1, S = 1, 2, ... \end{cases} \tag{46}$$

So the half-integer spin is different from the integer spin. In the half-integer spin we have the orthogral condition for the following state $|\Psi_k\rangle$, but not for integer spin.

Next we consider the state

$$\Psi_k = U_k\Psi_0 = \exp(ik\sum_n nS_n^z)\Psi_0 \tag{47}$$

where $k = 2\pi/N$.

We first prove that Ψ_k is orthogonal to the ground state:

$$\langle\Psi_0|\Psi_k\rangle = \langle\Psi_0|U_k|\Psi_0\rangle = \langle\Psi_0|TU_kT^{-1}|\Psi_0\rangle = -\langle\Psi_0|U_k|\Psi_0\rangle = -\langle\Psi_0|\Psi_k\rangle \tag{48}$$

where we used the condition for $\sum_n S_n^z = 0$ for the ground state. So it must be vanish, which is the orthogonal condition for Ψ_0 and Ψ_k.

$$\langle\Psi_0|\Psi_k\rangle = -\langle\Psi_0|\Psi_k\rangle = 0 \tag{49}$$

Next we calculate the energy difference between Ψ_k and Ψ_0. We need the following relations

$$U_k^{-1}S_n^xU_k = S_n^x\cos(kn) + S_n^y\sin(kn) \tag{50}$$

$$U_k^{-1}S_n^yU_k = -S_n^x\sin(kn) + S_n^y\cos(kn) \tag{51}$$

$$U_k^{-1}S_n^zU_k = S_n^z \tag{52}$$

which lead to

$$\begin{aligned} U_k^{-1}S_n^xS_{n+1}^xU_k &= (S_n^x\cos(kn) + S_n^y\sin(kn))(S_{n+1}^x\cos(k(n+1)) + S_{n+1}^y\sin(k(n+1))) \\ &= S_n^xS_{n+1}^x\cos(kn)\cos(k(n+1)) + S_n^yS_{n+1}^y\sin(kn)\sin(k(n+1)) \\ &+ S_n^yS_{n+1}^x\sin(kn)\cos(k(n+1)) + S_n^xS_{n+1}^y\cos(kn)\sin(k(n+1)) \end{aligned} \tag{53}$$

$$\begin{aligned} U_k^{-1}S_n^yS_{n+1}^yU_k &= (-S_n^x\sin(kn) + S_n^y\cos(kn))(-S_{n+1}^x\sin(k(n+1)) + S_{n+1}^y\cos(k(n+1))) \\ &= S_n^xS_{n+1}^x\sin(kn)\sin(k(n+1)) + S_n^yS_{n+1}^y\cos(kn)\cos(k(n+1)) \\ &- S_n^yS_{n+1}^x\cos(kn)\sin(k(n+1)) - S_n^xS_{n+1}^y\sin(kn)\cos(k(n+1)). \end{aligned} \tag{54}$$

We find

$$\begin{aligned}
&\langle\Psi_k|H|\Psi_k\rangle = \langle\Psi_0|U_k^{-1}HU_k|\Psi_0\rangle \\
&= \langle\Psi_0|H + (\cos(k)-1)\sum_n (S_n^x S_{n+1}^x + S_n^y S_{n+1}^y) + \sin(k)\sum_n (S_n^x S_{n+1}^y - S_n^y S_{n+1}^x)|\Psi_0\rangle \\
&= E_0 + (-\frac{1}{2}(\frac{2\pi}{N})^2 - O(N^{-4}))\langle\Psi_0|\sum_n (S_n^x S_{n+1}^x + S_n^y S_{n+1}^y)|\Psi_0\rangle + \sin(k)\langle\Psi_0|\sum_n (S_n^x S_{n+1}^y - S_n^y S_{n+1}^x)|\Psi_0\rangle \\
&\leq E_0 + \frac{2\pi^2}{N} \times const. \qquad (55)
\end{aligned}$$

The second term must have zero expectation in the ground state. The reason is as following. The definition of x, y could be switched, $S^x \to S^y, S^y \to S^x$. Under this rotation, the second term is odd, changing a sign, which means that it should vanish due to the symmetry reason. In the first term, the sum $\sum_n$ gives an order N, which cancels a factor of N in the final result.

To sum up, the energy gap vanishes if the chain is infinite long ($N \to \infty$).

Majumdar–Ghosh model ang valence-bond-solid

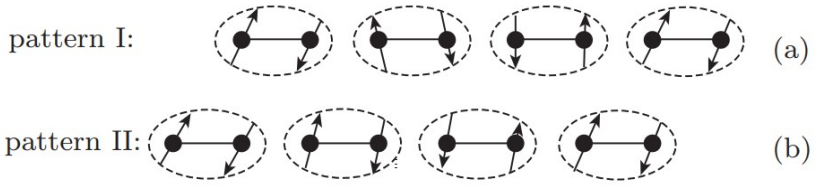
We introduce a second nearest-neighbor term in the spin chain model:

$$H^{MG} = J\sum_{i=1}^{N}(\mathbf{S}_i \cdot \mathbf{S}_{i+1} + \frac{1}{2}\mathbf{S}_i \cdot \mathbf{S}_{i+2}) \qquad (56)$$

Below we will show the ground state of this model is a dimer state or valence bond solid. The wave function is explicitly written as

$$|\psi_\pm\rangle = \prod_{n=1}^{N/2}(|\uparrow_{2n}\rangle|\downarrow_{2n\pm1}\rangle - |\downarrow_{2n}\rangle|\uparrow_{2n\pm1}\rangle)\frac{1}{\sqrt{2}} \qquad (57)$$

This is state is two-fold degenerated in 1D. The cartoon picture is as shown in Fig. **??**.



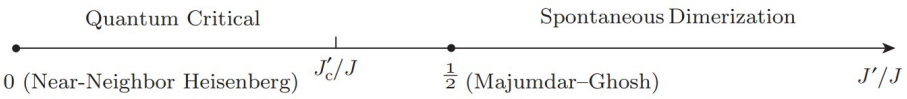


FIG. 2: Phase diagram of the $J - J'$ model, with nearest (J) and next-nearest (J') couplings of a spin-1/2 chain. The Majumdar–Ghosh (MG) model is a special point within the phase with spontaneous dimerization.

Let us rewrite the model as

$$H = \frac{J}{2}\sum_j (\mathbf{S}_i \cdot \mathbf{S}_{i+1} + \mathbf{S}_i \cdot \mathbf{S}_{i-1} + \mathbf{S}_{i-1} \cdot \mathbf{S}_{i+1})$$
$$= \frac{J}{4}\sum_j (\mathbf{S}_{i-1} + \mathbf{S}_{i+1} + \mathbf{S}_{i+1})^2 + const. \equiv= \frac{J}{4}\sum_i (\mathbf{J}_i)^2 \tag{58}$$

where the total spin of a triad of spins at sites i is

$$\mathbf{J}_i = \mathbf{S}_{i-1} + \mathbf{S}_{i+1} + \mathbf{S}_{i+1} \tag{59}$$

Its square has eigenvalues $J(J+1)$, where $J = 1/2, 3/2$.

The two exact ground states are made of nearest-neighbor pairs of spins forming singlets. The reason why they are exact ground states is the following. $\mathbf{J}_i$ is a sum of three-spin (in sequence) cluster terms, and the ground state of each term should have total spin-1/2 for the corresponding cluster. The two states of $\psi_\pm\rangle$ have the property that, among each cluster of three neighboring spins, two of them form a singlet, guaranteeing that the total spin of the cluster is 1/2; they are thus the ground state of every term $\mathbf{J}_i$.

These singlet bonds formed by nearestneighbor spins are often called valence bonds (in analogy to the chemical bonds that are often referred to by the same name in chemistry), and such spontaneously dimerized states are also called valence bond solid (VBS).

In the Majumdar–Ghosh model the Lieb–Schultz–Mattis theorem is satisfied in a very unusual way: the ground state has double degeneracy, and higher exciations are gapped.

AKLT model

In this section, we present detailed derivation of analytical results for AKLT model. We will demonstrate that, creating a boundary by cutting a valence bond in VBS state will

lead to a fractional spin$-1/2$ forms near the boundary. We found that, despite the spatial oscillation due to the anti-ferromagnetic exchange, the spin magnetization exponentially decays to zero away from the boundary, which makes it available to define a edge spin localized in the vicinity of the boundary. Importantly, the net spin magnetization of edge spin is always quantized to fractional value.

We start from the well-known AKLT model [AKLT 1988], where the analytical solution will be important for the numerical results later on. The AKLT model placed on a open chain with N sites can be described by

$$H_{AKLT} = \sum_{i=1}^{N-1}(\mathbf{S}_i \cdot \mathbf{S}_{i+1} + \frac{1}{3}(\mathbf{S}_i \cdot \mathbf{S}_{i+1})^2 + \frac{2}{3}) = \sum_{i=1}^{N-1} P_{i,i+1}^{J=2} \tag{60}$$

where $\mathbf{S}$ is spin-1 operator. Each term $P_{i,i+1}^{J=2}$ projects the bond spin $\mathbf{J}_{i,i+1} = \mathbf{S}_i + \mathbf{S}_{i+1}$ onto the subspace of magnitude $J = 2$.

$$P_{i,i+1}^{J=2} \equiv \frac{1}{2}(\mathbf{S}_i \cdot \mathbf{S}_{i+1}) + \frac{1}{6}(\mathbf{S}_i \cdot \mathbf{S}_{i+1})^2 + \frac{1}{3} = \frac{1}{24} J_{i,i+1}^2 (J_{i,i+1}^2 - 2) \tag{61}$$

This operator annihilates total spin zero or one states, and gives unity on spin two states. If we introduce the basis of total spin states $|s_t, m_t\rangle$, where s_t is the total spin quantum number of $S_i + S_{i+1}$:

$$P^{J=2} = \sum_{m_t=-s_t,s_t} |2, m_t\rangle\langle 2, m_t| \tag{62}$$

Since Eq. 60 equivalent to spin projection operator, the exact zero energy ground state can be constructed accordingly. Here we utilize the Schwinger boson representation to express spin operators as $\hat{S}_j^+ = a_j^\dagger b_j$, $\hat{S}_j^z = \frac{1}{2}(a_j^\dagger a_j - b_j^\dagger b_j)$, where $a^\dagger$ and $b^\dagger$ satisfy the commutation relations $[a_j, a_j^\dagger] = [b_j, b_j^\dagger] = \delta_{ij}$ [Arovas 1988]. To reproduce the dimension of spin-1 Hilbert space at each site, we should impose the constraint that the total boson occupation number $a_j^\dagger a_j + b^\dagger b_j = 2$. With the help of Schwinger boson representation, the(un-normalized) ground state of Eq. 60 corresponds to a valence bond solid (VBS) of adjacent dimers (Fig. 3):

$$\begin{aligned} &|\Phi_N(\alpha, \beta)\rangle = (a_1^\dagger)^{\frac{1}{2}+\alpha}(b_1^\dagger)^{\frac{1}{2}-\alpha} \otimes \\ &\prod_{i=1}^{N-1}(a_i^\dagger b_{i+1}^\dagger - b_i^\dagger a_{i+1}^\dagger) \otimes (a_N^\dagger)^{\frac{1}{2}+\beta}(b_N^\dagger)^{\frac{1}{2}-\beta}|0\rangle, \end{aligned} \tag{63}$$

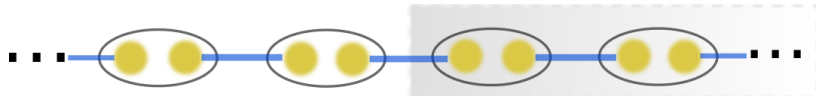


FIG. 3: Schematic plot of VBS as the exact ground state of spin-1 AKLT model. Each original spin-1 (black circle) is written as two spin$-1/2$ (yellow dot) in a triplet state. The ground state is then obtained by tensor product of singlet bonds (blue line) connecting nearest-neighbor adjacent spin$-1/2$, thus forming a crystalline pattern of valence bonds. Bipartitioning the chain into left and right part inevitably cuts one of valence bond, therefore two disentangled spin$-1/2$ spins form in the vicinity of the virtual boundary at the cut position.

where α, β can be chosen as $\pm 1/2$, representing the free edge degree of freedom thus leading to four-fold degenerate ground states.

Following the discussion above, we explicitly consider the ground state of AKLT model in open boundary condition:

$$|\Phi_N(\alpha,\beta)\rangle = (a_1^\dagger)^{\frac{1}{2}+\alpha}(b_1^\dagger)^{\frac{1}{2}-\alpha}\prod_{i=1}^{N-1}(a_i^\dagger b_{i+1}^\dagger - b_i^\dagger a_{i+1}^\dagger)(a_N^\dagger)^{\frac{1}{2}+\beta}(b_N^\dagger)^{\frac{1}{2}-\beta}|0\rangle, \tag{64}$$

where α, β can be chosen as $\pm 1/2$, representing four-fold degenerate ground states with different boundary condition at site 1 and N. Using this wavefunction it is now straightforward to evaluate the magnetization on each site of the open chain. Next we perform the calculation on $|\Phi(\alpha = 1/2, \beta = 1/2)\rangle$ and the results for the other three ground states are similar.

First we noticed that $|\Phi\rangle$ is unnormalized so let us determine the normalization factor

$\langle\Phi|\Phi\rangle$:

$$\langle\Phi(\frac{1}{2},\frac{1}{2})|\Phi(\frac{1}{2},\frac{1}{2})\rangle = \langle 0|a_N\prod_{i=1}^{N-1}(a_ib_{i+1}-b_ia_{i+1})a_1a_1^\dagger\prod_{i=1}^{N-1}(a_i^\dagger b_{i+1}^\dagger - b_i^\dagger a_{i+1}^\dagger)a_N^\dagger|0\rangle$$

$$= \prod_{i=1}^{N}[\int\frac{2S+1}{4\pi}d\Omega_i]|u_1|^2|u_N|^2\prod_{i=1}^{N-1}[\frac{1-\Omega_i\cdot\Omega_{i+1}}{2}]$$

$$= \prod_{i=1}^{N}[\int\frac{2S+1}{4\pi}d\Omega_i]|u_1|^2|u_N|^2\times[4\pi\sum_{l_1=0}^{\infty}\frac{C_{l_1}}{2l_1+1}\sum_{m_1=-l_1}^{l_1}Y_{l_1,m_1}(\Omega_1)Y^*_{l_1,m_1}(\Omega_2)]$$

$$\times[4\pi\sum_{l_2=0}^{\infty}\frac{C_{l_2}}{2l_2+1}\sum_{m_2=-l_2}^{l_2}Y_{l_2,m_2}(\Omega_2)Y^*_{l_2,m_2}(\Omega_3)]$$

$$...\times[4\pi\sum_{l_{N-1}=0}^{\infty}\frac{C_{l_{N-1}}}{2l_{N-1}+1}\sum_{m_{N-1}=-l_{N-1}}^{l_{N-1}}Y_{l_{N-1},m_{N-1}}(\Omega_{N-1})Y^*_{l_{N-1},m_{N-1}}(\Omega_N)]$$

$$= \frac{(2S+1)^N}{4\pi}\sum_{l_1,l_2,...,l_{N-1}}\frac{C_{l_1}}{2l_1+1}\frac{C_{l_2}}{2l_2+1}...\frac{C_{l_{N-1}}}{2l_{N-1}+1}\delta_{l_1,l_2}\delta_{l_2,l_3}...\delta_{l_{N-2},l_{N-1}}$$

$$= (2S+1)^N[C_0^{N-1}\frac{1}{4}+(\frac{C_1}{3})^{N-1}\frac{1}{4}\frac{1}{3}] = (2S+1)^N[\frac{1}{2^{N+1}}+(-)^{N-1}\frac{1}{2^{N+1}}\frac{1}{3^N}]$$

where we used

$$\int d\Omega\cos^2\frac{\theta}{2}Y_{l,m}(\Omega) = \int d\Omega\frac{1+\cos\theta}{2}Y_{l,m}(\Omega) = \frac{\sqrt{4\pi}}{2}\delta_{l=0}\delta_{m=0}+\frac{1}{2}\sqrt{\frac{4\pi}{3}}\delta_{l=1}\delta_{m=0} \qquad (65)$$

Next, with the help of the relation $\hat{S} = \int\frac{(S+1)(2S+1)}{4\pi}\hat{\Omega}|\Omega\rangle\langle\Omega|$, we calculate the magnetization at the site 1 (the last site N is equivalent):

$$\langle\Phi(\frac{1}{2},\frac{1}{2})|S_1^z|\Phi(\frac{1}{2},\frac{1}{2})\rangle = \langle\Phi(\frac{1}{2},\frac{1}{2})|\int\frac{(S+1)(2S+1)}{4\pi}\Omega_1^z|\Omega_1\rangle\langle\Omega_1||\Phi(\frac{1}{2},\frac{1}{2})\rangle$$

$$= (S+1)\prod_{i=1}^{N}[\int\frac{(2S+1)}{4\pi}\Omega_i^z]|u_1|^2|u_N|^2\Omega_1\prod_{i=1}^{N-1}[\frac{1-\Omega_i\cdot\Omega_{i+1}}{2}]$$

$$= (S+1)\prod_{i=1}^{N}[\frac{2S+1}{4\pi}d\Omega_i]|u_1|^2|u_N|^2\Omega_1[4\pi\sum_{l_1=0}^{\infty}\frac{C_{l_1}}{2l_1+1}\sum_{m_1=-l_1}^{l_1}Y_{l_1,m_1}(\Omega_1)Y^*_{l_1,m_1}(\Omega_2)]$$

$$\times[4\pi\sum_{l_2=0}^{\infty}\frac{C_{l_2}}{2l_2+1}\sum_{m_2=-l_2}^{l_2}Y_{l_2,m_2}(\Omega_2)Y^*_{l_2,m_2}(\Omega_3)]$$

$$...\times[4\pi\sum_{l_{N-1}=0}^{\infty}\frac{C_{l_{N-1}}}{2l_{N-1}+1}\sum_{m_{N-1}=-l_{N-1}}^{l_{N-1}}Y_{l_{N-1},m_{N-1}}(\Omega_{N-1})Y^*_{l_{N-1},m_{N-1}}(\Omega_N)]$$

$$= (S+1)\frac{(2S+1)^N}{4\pi}\sum_{l_1,l_2,...,l_{N-1}}\frac{C_{l_1}}{2l_1+1}\frac{C_{l_2}}{2l_2+1}...\frac{C_{l_{N-1}}}{2l_{N-1}+1}\delta_{l_1,l_2}\delta_{l_2,l_3}...\delta_{l_{N-2},l_{N-1}}$$

$$= (S+1)(2S+1)^N[\frac{1}{2^{N+1}}\frac{1}{3}+(-)^{N-1}\frac{1}{2^{N+1}}\frac{1}{3^N}] = (2S+1)^N[\frac{1}{2^{N+1}}\frac{2}{3}+2(-)^{N-1}\frac{1}{2^{N+1}}\frac{1}{3^N}]$$

where we used the relations Eq. 65 and

$$\int d\Omega \cos^2\frac{\theta}{2}\cos\theta Y_{l,m}(\Omega) = \int d\Omega \frac{1+\sqrt{\frac{4\pi}{3}}Y_{10}(\Omega)}{2}\sqrt{\frac{4\pi}{3}}Y_{10}(\Omega)Y_{l,m}(\Omega)$$
$$= \frac{1}{2}[\sqrt{\frac{4\pi}{3}}\delta_{l=1}\delta_{m=0} + \sqrt{\frac{4\pi}{3}}\sqrt{\frac{4\pi}{3}}\sqrt{\frac{3*3*(2l+1)}{4\pi}}\begin{pmatrix}1&1&l\\0&0&0\end{pmatrix}\begin{pmatrix}1&1&l\\0&0&m\end{pmatrix}]$$
$$= \frac{1}{2}[\sqrt{\frac{4\pi}{3}}\delta_{l=1}\delta_{m=0} + \frac{\sqrt{4\pi}}{3}\delta_{l=0}\delta_{m=0} + (l\geq 2)] \tag{66}$$

Finally, we have

$$\langle S_1^z\rangle = \frac{\langle\Phi(\frac{1}{2},\frac{1}{2})|S_1^z|\Phi(\frac{1}{2},\frac{1}{2})\rangle}{\langle\Phi(\frac{1}{2},\frac{1}{2})|\Phi(\frac{1}{2},\frac{1}{2})\rangle} = \frac{(2S+1)^N[\frac{1}{2^{N+1}}\frac{2}{3}+2(-)^{N-1}\frac{1}{2^{N+1}}\frac{1}{3^N}]}{(2S+1)^N[\frac{1}{2^{N+1}}+(-)^{N-1}\frac{1}{2^{N+1}}\frac{1}{3^N}]}$$
$$= \frac{\frac{2}{3}+2(-)^{N-1}\frac{1}{3^N}}{1+(-)^{N-1}\frac{1}{3^N}} \tag{67}$$

Furthermore, we can proof the general expression for any site i:

$$\langle S_i^z\rangle = (-)^{i-1}\frac{\frac{2}{3}(\frac{1}{3})^{i-1}+2(-)^{N-1}\frac{1}{3^N}3^{i-1}}{1+(-)^{N-1}\frac{1}{3^N}} \tag{68}$$

Hence, in the thermodynamic limit we have two important observations. First, the spin magnetization has the real space distribution as

$$\langle S_i^z\rangle \to \frac{2}{3}(-\frac{1}{3})^{i-1} = (-)^{i-1}\frac{2}{3}e^{-(i-1)\ln 3} \tag{69}$$

This explicitly shows that the spin magnetization is decreasing exponentially away from the boundary of the chain with a length scale equal to $\xi = \ln 3$, which is consistent with the correlation length in the bulk (using periodic boundary condition). This leads to the picture that emergent spin is localized near the open boundary therefore we can define the emergent spin near the boundary as the edge spin. Second, the total net spin magnetization of edge spin is

$$\Delta S^z|_{edge} = \lim_{N\to\infty}\sum_{i\in edge}\langle S_i^z\rangle \to \sum_i \frac{2}{3}(-\frac{1}{3})^{i-1} = \frac{1}{2} \tag{70}$$

Spin coherent state.— In order to calculate the quantities using AKLT wavefunction, it is convenient to introduce the concept of spin coherent state [Arovas1988,Radcliffe1971,Arecchi1972]. To be specific, let us introduce the spin coherent state wavefunction here:

$$|\hat{\Omega}_i\rangle = \frac{u_i a_i^\dagger + v_i b_i^\dagger}{\sqrt{(2S)!}}|0\rangle, \tag{71}$$

and for a point $\Omega_i = (\theta_i, \phi_i)$ on the unit sphere we define the parameters

$$u_i = \cos\frac{\theta_i}{2}e^{i\frac{\phi_i}{2}}, v_i = \sin\frac{\theta_i}{2}e^{-i\frac{\phi_i}{2}}. \tag{72}$$

Here we have already fixed the $U(1)$ gauge degree of freedom since it has no physical content.

The spin coherent state defined this way satisfies several useful conditions, such as

$$\hat{\Omega}\cdot\hat{S}|\hat{\Omega}\rangle = S|\hat{\Omega}\rangle, \tag{73}$$

$$\langle 0|a^{S+l}b^{S-l}|\hat{\Omega}\rangle = \sqrt{(2S)!}u^{s+l}v^{S-l}, \tag{74}$$

and the completion relation

$$\frac{(2S+1)d\hat{\Omega}}{4\pi}|\Omega\rangle\langle\Omega| = 1. \tag{75}$$

Taking the advantage of spin coherent representation, we can calculate the quantities using the ground state of AKLT model. For example, the normalization factor of the ground state of AKLT model is obtained by:

$$\begin{aligned} N_0 &= \langle VBS|VBS\rangle = \langle VBS|\prod_i[\int\frac{(2S+1)d\hat{\Omega}_i}{4\pi}]|\Omega_i\rangle\langle\Omega_i||VBS\rangle \\ &= \prod_i\int\frac{(2S+1)d\hat{\Omega}_i}{4\pi}|u_iv_{i+1}-v_iu_{i+1}|^2 \\ &= \prod_i\int\frac{(2S+1)d\hat{\Omega}_i}{4\pi}[\frac{1-\hat{\Omega}_i\cdot\hat{\Omega}_{i+1}}{2}]. \end{aligned} \tag{76}$$

To explicitly complete the integral, we need the relation

$$\begin{aligned} &\frac{1-\hat{\Omega}_i\cdot\hat{\Omega}_j}{2} = \frac{1}{2}-\frac{1}{4}(\Omega_i^\dagger\Omega_j^- + \Omega_i^-\Omega_j^\dagger) + \frac{1}{2}\Omega_i^z\Omega_j^z \\ &= 4\pi[\frac{1}{2}Y_{00}(\hat{\Omega}_i)Y_{00}^*(\hat{\Omega}_j) - \frac{1}{6}Y_{1,-1}(\hat{\Omega}_i)Y_{1,-1}^*(\hat{\Omega}_j) - \frac{1}{6}Y_{1,0}(\hat{\Omega}_i)Y_{1,0}^*(\hat{\Omega}_j) - \frac{1}{6}Y_{1,1}(\hat{\Omega}_i)Y_{1,1}^*(\hat{\Omega}_j)] \\ &= 4\pi\sum_{l=0}^{\infty}\frac{C_l}{2l+1}\sum_{m=-l}^{l}Y_{l,m}(\hat{\Omega}_i)Y_{l,m}^*(\hat{\Omega}_j) \end{aligned}$$

and

$$C_0 = \frac{1}{2}, C_1 = -\frac{1}{2}, C_{k>1} = 0. \tag{77}$$

Here we also introduced the spherical harmonics function $Y_{l,m}(\mathbf{\Omega})$:

$$Y_{0,0}(\mathbf{\Omega}) = \frac{1}{\sqrt{4\pi}}, Y_{1,-1}(\mathbf{\Omega}) = \sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi}, Y_{1,0}(\mathbf{\Omega}) = \sqrt{\frac{3}{4\pi}}\cos\theta, Y_{1,1}(\mathbf{\Omega}) = -\sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\phi} \tag{78}$$

In the integral calculation, the orthogonal relation will be very helpful:

$$\int d\hat{n} Y_{l,m}(\hat{\Omega}) Y^*_{l',m'}(\hat{\Omega}) = \delta_{l,l'}\delta_{m,m'} \tag{79}$$

Haldane conjecture: O(3) non-linear sigma model with topological $\theta-$term

For spin-1 Heisenberg chain, Haldane conjectured the ground state is gapped [Haldane, 1983]. He also proposed that an excitation gap exists not only for spin 1, but also for all antiferromagnetic integer spin chains.

For spin-1 chain, one can deduce the Heisenberg point is smoothly connected to the AKLT point (they are in the same phase actually). But AKLT model appeared in 1987, how did Haldane get his proposal? We will review his 1983 seminar paper below.

Spin berry phase.— Imagine now that we want to compute the partition function

$$\begin{aligned} Z &= Tr[e^{-\beta H}] = Tr[\lim_{N\to\infty}(e^{-\delta t H})^N] \\ &= \lim_{N\to\infty}(\frac{2S+1}{4\pi})^N(\prod_{a=1}^{N}\int d^3 n_a \delta(\mathbf{n}^2-1)\langle \mathbf{n}(t_a)|e^{-\delta t H}|\mathbf{n}(t_{a+1})\rangle) \end{aligned} \tag{80}$$

Here we take the evolution of the system in imaginary time with periodic boundary conditions. We can decompose the evolution in imaginary time into N infinitesimal steps of length δt, using the Trotter expansion.

Keeping up to the first order in δt, we have

$$\begin{aligned} \langle \mathbf{n}(t_a)|e^{-\delta t H}|\mathbf{n}(t_{a+1})\rangle &= \langle \mathbf{n}(t_a)|\mathbf{n}(t_{a+1})\rangle - \delta t\langle \mathbf{n}(t_a)|H|\mathbf{n}(t_a)\rangle + O(\delta t^2) \\ &\approx \langle \mathbf{n}(t_a)(|\mathbf{n}(t_a)\rangle + \delta t\frac{d|\mathbf{n}(t_{a+1})\rangle}{dt}) - \delta t\langle \mathbf{n}(t_a)|H|\mathbf{n}(t_a)\rangle + O(\delta t^2) \\ &= 1 + \delta t[\langle \mathbf{n}(t_a)|\dot{\mathbf{n}}(t_a)\rangle - \langle \mathbf{n}(t_a)|H|\mathbf{n}(t_a)\rangle] + O(\delta t^2) \\ &\approx \exp[\delta t[\langle \mathbf{n}(t_a)|\dot{\mathbf{n}}(t_a)\rangle - \langle \mathbf{n}(t_a)|H|\mathbf{n}(t_a)\rangle]] \end{aligned} \tag{81}$$

We can write down the partition function as

$$Z \equiv \int D[\mathbf{n}] e^{-S[\mathbf{n}]}, \tag{82}$$

$$S[\mathbf{n}] = -iS \int_0^\beta d\tau \dot{\mathbf{n}} \cdot \mathbf{A}[\mathbf{n}] + \int_0^\beta dt \langle \mathbf{n}(t)|H|\mathbf{n}(t)\rangle + O(\delta t^2) \tag{83}$$

$$= -iS\omega[\mathbf{n}(t)] + \int_0^\beta dt \langle \mathbf{n}(t)|H|\mathbf{n}(t)\rangle + O(\delta t^2) \tag{84}$$

where $\omega[\mathbf{n}(t)]$ is the total berry phase (solid angle), or the area in the unit sphere bounded by the curve $\mathbf{n}(t)$. The berry connection is defined as

$$\dot{\mathbf{n}} \cdot \mathbf{A}[\mathbf{n}] = \langle \mathbf{n}(t)|\dot{\mathbf{n}}(t)\rangle, \;\; \mathbf{A}[\mathbf{n}] = i\langle \mathbf{n}|\nabla_{\mathbf{n}}|\mathbf{n}\rangle \tag{85}$$

Berry phase.— Assuming a physical system is depended on some parameters $\mathbf{R} = (R_1, R_2, \cdots, R_N)$, we have the snapshot Hamiltonian $H(\mathbf{R})$, its eigen-values and eigen-states:

$$H(\mathbf{R})|n(\mathbf{R})\rangle = E_n(\mathbf{R})|n(\mathbf{R})\rangle \tag{86}$$

where $|n(\mathbf{R})\rangle$ can have an arbitrary phase prefactor.

The parameters $\mathbf{R}(\mathbf{t})$ are slowly changed with time t, then the *adiabatic* evolution of time-dependent Schrodinger equation:

$$i\frac{d}{dt}|\psi(t)\rangle = H(\mathbf{R}(\mathbf{t}))|\psi(t)\rangle \tag{87}$$

Taking the Ansatz $|\psi(t)\rangle = e^{i\omega_n(t)} e^{-i\int_0^t E_n(\mathbf{R}(\mathbf{t}'))dt'} |n(\mathbf{R}(\mathbf{t}))\rangle$, we have

$$i\frac{d}{dt}|\psi(t)\rangle = e^{i\omega_n(t)} e^{-i\int_0^t E_n(\mathbf{R}(\mathbf{t}'))dt'} \left[-\frac{d\omega_n(t)}{dt}|n(\mathbf{R}(\mathbf{t}))\rangle + E_n(R(t))|n(\mathbf{R}(\mathbf{t}))\rangle + i|\frac{d}{dt}n(\mathbf{R}(t))\rangle \right] \tag{88}$$

Here $e^{-i\int_0^t E_n(\mathbf{R}(\mathbf{t}'))dt'}$ is the dynamic phase, and ω_n is the geometric phase which will be clarified below.

We insert the above Ansatz into the rhs of the Schrödinger equation and use the "adiabatic" condition $H(R(t))|\psi(t)\rangle = E_n(R(t))|\psi(t)\rangle$,

$$-\left(\frac{d}{dt}\omega_n\right)|n\rangle + i\left|\frac{d}{dt}n\right\rangle = 0 \tag{89}$$

Multiply from the left by $\langle n(R(t))|$, and obtain the Berry phase expression:

$$\frac{d}{dt}\omega_n = i\langle n(R(t))|\frac{d}{dt}|n(R(t))\rangle = i\frac{dR(t)}{dt}\langle n(R)|\bigtriangledown_R|n(R)\rangle \tag{90}$$

$$\omega_n(\mathcal{C}) = i\int_{\mathcal{C}} dt\frac{dR(t)}{dt}\langle n(\mathbf{R})|\nabla_{\mathbf{R}}n(\mathbf{R})\rangle = \int_{\mathcal{C}} i\langle n(\mathbf{R})|\nabla_{\mathbf{R}}n(\mathbf{R})\rangle d\mathbf{R} \tag{91}$$

We can define Berry connection:

$$\mathbf{A}^{(n)}(\mathbf{R}) = i\langle n(\mathbf{R})|\nabla_{\mathbf{R}}n(\mathbf{R})\rangle = -Im\langle n(\mathbf{R})|\nabla_{\mathbf{R}}n(\mathbf{R})\rangle \tag{92}$$

Gauge transformation $|n(\mathbf{R})\rangle \to e^{i\alpha(\mathbf{R})}|n(\mathbf{R})\rangle$, $\mathbf{A}^{(n)}(\mathbf{R}) \to \mathbf{A}^{(n)}(\mathbf{R}) - \nabla_{\mathbf{R}}\alpha(\mathbf{R})$ and $\omega = \oint \mathbf{A}(\mathbf{R})d\mathbf{R}$ is gauge invariant.

It is better to give an example here: A spin-1/2 in the presence of time-dependent magnetic field

$$H = -\mathbf{B}(t)\cdot\mathbf{S} = -[\sigma_x X(t) + \sigma_y Y(t) + \sigma_z Z(t)] \tag{93}$$

We will let the direction of B in space be the control parameter of the Hamiltonian: $\mathbf{R}(t) = (X(t), Y(t), Z(t)), R = |\mathbf{R}| = \sqrt{X^2+Y^2+Z^2}$.

The instantaneous eigenvalue is $|-\sigma\cdot\mathbf{R}(t) - E| = 0$, thus $E_m = mR(t), m = \pm$. The Berry phase is calculated below. First, we have $\nabla_{R(t)}H(R) = -\sigma_x\hat{e}_x - \sigma_y\hat{e}_y - \sigma_z\hat{e}_z$. Second, we calculate the element by choosing the direction of R along z-direction. $\sigma_z|\pm\rangle = \pm|\pm\rangle$, $\sigma_x|\pm\rangle = |-+\rangle$, $\sigma_y|\pm\rangle = \pm i|-+\rangle$. Third, $\langle+|\nabla_R H|-\rangle = \hat{e}_x + i\hat{e}_y$, and $\langle-|\nabla_R H|+\rangle = \hat{e}_x - i\hat{e}_y$. $\langle+|\nabla_R H|-\rangle \times \langle-|\nabla_R H|+\rangle = -2i\hat{e}_z$. Thus we have $\mathbf{B}^{(+)} = -Im\sum_{n'\neq n}\frac{\langle n|\nabla H|n'\rangle\times\langle n'|\nabla H|n\rangle}{(E_n - E_{n'})^2} = \frac{\hat{e}_z}{2R^2}$, and $\mathbf{B}^{-} = -\frac{\hat{e}_z}{2R^2}$. In general, for arbitrary direction $\vec{R}$, we have

$$\mathbf{B}^{(+)} = \frac{\vec{R}}{2R^3}, \mathbf{B}^{(-)} = -\frac{\vec{R}}{2R^3} \tag{94}$$

The resulting berry phase for a circle-like loop on the sphere is

$$\omega_n = \int_{\mathcal{C}} \mathbf{B}^{(\pm)}\cdot dS = \pm\int_0^{2\pi} d\phi\int_0^{\theta} d\theta\sin\theta R\vec{R}\frac{\vec{R}}{2R^3} = \pm\pi(1-\cos\theta) = \pm\frac{\Omega(C)}{2} \tag{95}$$

where Ω is the solid angular enclosed by the closed path $\mathcal{C}$. Since the berry phase is related the geometric path, we also call it as geometric phase.

Note that $\mathbf{B}^{(+)} = \frac{\vec{R}}{2R^3}, \mathbf{B}^{(-)} = -\frac{\vec{R}}{2R^3}$ implies the magnetic monopole! This is the same formula we obtained earlier from the eigenstate formalism and is the flux through an area bounded by the curve C of a monopole with strength $\pm 1/2$ located at the degeneracy. The Berry curvature, which is gauge invariant, is singular at he origin $R = 0$ (but regular everywhere else).

If a monopole of strength e_M exists,, i.e. $\partial \cdot \mathbf{B} = 4\pi e_M \delta(R)$, then the magnetic field around it is $\mathbf{B} = e_M \frac{\mathbf{R}}{R^3}$, similar as the electric field generated by a charge. But electric field is related to the potential as $E = -\partial V$, it is impossible to find a regular-everywhere vector potienal that $\mathbf{B} = \partial \times \mathbf{A}$. To proof this, let us consider a sphere S^2 of radium $R = 1$ enclosing the monopole, and denote $\Sigma_{N(S)}$ as the north and south hemisphere, which meet at the equator C. Next we have $\int_{\Sigma_N} \mathbf{B} \cdot \mathbf{n} dS = \int_C \mathbf{A} \cdot dl$, and $\int_{\Sigma_S} \mathbf{B} \cdot \mathbf{n} dS = \int_{-C} \mathbf{A} \cdot dl$. Summing together the two fluxes we conclude that $\int_{S^2} \mathbf{B} \cdot \mathbf{n} dS = \int_C \mathbf{A} \cdot dl + \int_{-C} \mathbf{A} \cdot dl = 0$. It is contrary to the fact that the flux should be $4\pi e_M$.

Again, another way of appreciating the singularity of vector potential, we can consider a small circle C, of radius $R \sin\theta$, encircling the north pole of spehere at an angle θ. We already know the magnetic flux through the solid angle $2\pi(1 - \cos\theta)$ enclosed by C is $e_M 2\pi(1 - \cos\theta)$. We can represent such a flux by the line integral of a vector potential $\mathbf{A}$ as

$$\int \mathbf{A} \cdot dl = A_\phi 2\pi R \sin\theta = e_M 2\pi(1 - \cos\theta), \rightarrow A_\phi = \frac{e_M}{R} \frac{1 - \cos\theta}{\sin\theta} \tag{96}$$

which becomes singlular at θ.

We consider a one-dimensional antiferromagnetic Heisenberg spin chain:

$$H = J \sum_i \mathbf{S}_i \cdot \mathbf{S}_{i+1}, \tag{97}$$

with $J > 0$. Using the spin coherent state, we have the effective action as

$$S[\{\mathbf{n}_i\}] = -iS \sum_i \omega[\{\mathbf{n}_i\}] + JS^2 \int_0^\beta dt \sum_i \langle \mathbf{n}_i(t)|H|\mathbf{n}_i(t)\rangle \tag{98}$$

In order to take the continuum limit in the spatial direction, we need to identify the low energy, large scale degrees of freedom that can be considered as slowly varying fields in the

action. We can then write the ansatz

$$\mathbf{n}_i = (-1)^i \mathbf{m}_i + a\mathbf{L}_i \tag{99}$$

Here $\mathbf{m}_i$ is the unit vector describing the magnetization $|\mathbf{m}_i| = 1$. In order to satisfy the normalization condition $|\mathbf{n}_i| = 1$, we require that, up to the order a

$$\mathbf{m}_i \cdot \mathbf{L}_i = 0. \tag{100}$$

Here a the lattice spacing constant. The field $\mathbf{L}_i$, playing the role of angular momentum for $\mathbf{n}_i$ is chosen to have dimension of density and is responsible of a net magnetization which is supposed to be small. The constant appearing in the front of $\mathbf{L}_i$ is due to the physical unit, which will become clear in the latter.

Using this ansatz, we have

$$\begin{aligned}
H &= JS^2 \sum_i [(-1)^i \mathbf{m}_i + a\mathbf{L}_i][(-1)^{i+1}\mathbf{m}_{i+1} + a\mathbf{L}_{i+1}] \\
&= -JS^2 \sum_i \mathbf{m}_i \cdot \mathbf{m}_{i+1} + JS^2 a^2 \sum_i \mathbf{L}_i \cdot \mathbf{L}_{i+1} + JS^2 a \sum_i (-1)^i [\mathbf{m}_i \cdot \mathbf{L}_{i+1} + \mathbf{m}_i \cdot \mathbf{L}_{i-1}] \\
&= -JS^2 \sum_i [1 + a\mathbf{m}_i \cdot \frac{\partial \mathbf{m}_i}{\partial x} + \frac{a^2}{2}\mathbf{m}_i \cdot \frac{\partial^2 \mathbf{m}_i}{\partial x^2}] + JS^2 a^2 \sum_i \mathbf{L}_i^2 + 2JS^2 a^2 \sum_i (-1)^i \mathbf{m}_i \cdot \mathbf{L}_i + O(a^3) \\
&= -JS^2 \sum_i [1 + \frac{a^2}{2}\mathbf{m}_i \cdot \nabla^2 \mathbf{m}_i] + JS^2 a^2 \sum_i \mathbf{L}_i^2 + O(a^3)
\end{aligned} \tag{101}$$

where we used $\mathbf{n} \cdot \partial_x \mathbf{n} = 0$ and $\mathbf{m}_i \cdot \mathbf{L}_i = 0$.

Next we turn to the Berry phase.

$$\begin{aligned}
\sum_i \omega[\{\mathbf{n}_i\}] &= \sum_i \omega[\{(-1)^i \mathbf{m}_i + a\mathbf{L}_i\}] \\
&\approx \sum_i \omega[\{(-1)^i \mathbf{m}_i\}] + \int_0^\beta d\tau \frac{\delta \omega[\{\mathbf{n}_i\}]}{\delta \mathbf{n}(\tau)} \cdot a\mathbf{L}_i
\end{aligned} \tag{102}$$

There are two terms here. We can consider them one by one.

First, since the berry phase is related to the solid angle made of three spins, the solid angle is odd under $\mathbf{n} \to -\mathbf{n}$, we have

$$\omega[\{-\mathbf{n}_i\}] = -\omega[\{\mathbf{n}_i\}] \tag{103}$$

On the one-dimensional lattice, we have

$$\begin{aligned}\sum_{i=1}^{L}(-1)^i\omega[\{\mathbf{m}_i\}] &= \sum_{i=1}^{L/2}\omega[\{\mathbf{m}_{2i}\}]-\omega[\{\mathbf{m}_{2i-1}\}] \\ &= \frac{1}{2}\int_0^\beta d\tau\int dx\frac{\partial\omega[\{\mathbf{m}(x,\tau)\}]}{\partial x} = \frac{1}{2}\int_0^\beta d\tau\int dx\frac{\delta\omega[\{\mathbf{m}(x,\tau)\}]}{\delta\mathbf{m}(x,\tau)}\frac{\partial\mathbf{m}(x,\tau)}{\partial x} \\ &= \frac{1}{2}\int_0^\beta d\tau\int dx[\frac{\partial\mathbf{m}(x,\tau)}{\partial\tau}\times\mathbf{m}(x,\tau)]\cdot\frac{\partial\mathbf{m}(x,\tau)}{\partial x} \end{aligned} \tag{104}$$

The factor of 1/2 appears because two adjacent sites are treated together.

In the last line, owing to $\omega[\{\mathbf{n}\}]\sim\int_0^\beta d\tau\dot{\mathbf{n}}(\tau)\cdot\mathbf{A}$,

$$\begin{aligned}\delta\omega[\{\mathbf{n}\}] &= \int_0^\beta d\tau\delta(\dot{\mathbf{n}}(\tau)\cdot\mathbf{A}) \\ &= \int_0^\beta d\tau[\frac{\partial A^\alpha}{\partial n^\beta}\delta n^\beta\dot{n}^\alpha + A^\alpha\frac{\partial}{\partial t}\delta n^\alpha] \\ &= \int_0^\beta d\tau[\frac{\partial A^\alpha}{\partial n^\beta}\delta n^\beta\dot{n}^\alpha + A^\alpha\frac{\partial}{\partial t}\delta n^\alpha] + [\frac{\partial A^\alpha}{\partial n^\beta}\dot{n}^\beta\delta n^\alpha - \frac{\partial A^\alpha}{\partial n^\beta}\dot{n}^\beta\delta n^\alpha] \\ &= \int_0^\beta d\tau\frac{\partial A^\alpha}{\partial n^\beta}\epsilon^{\alpha\beta\gamma}(\dot{\mathbf{n}}\times\delta\mathbf{n})_\gamma + \int_0^\beta d\tau\frac{d}{d\tau}(\mathbf{A}\cdot\delta\mathbf{n}) \\ &= \int_0^\beta d\tau\mathbf{n}\cdot(\dot{\mathbf{n}}\times\delta\mathbf{n}) = \int_0^\beta d\tau\delta\mathbf{n}\cdot(\mathbf{n}\times\dot{\mathbf{n}}) \end{aligned} \tag{105}$$

Here we have used the condition (from Stokes theorem): $(\nabla\times\mathbf{A})\cdot\mathbf{n}=1=\epsilon^{\alpha\beta\gamma}\frac{\partial A^\alpha}{\partial n^\beta}n^\gamma$, so $\epsilon^{\alpha\beta\gamma}\frac{\partial A^\alpha}{\partial n^\beta}=n^\gamma$. Then we get the result

$$\frac{\delta\omega[\{\mathbf{n}\}]}{\delta\mathbf{n}(\tau)} = \int_0^\beta d\tau\frac{\partial\mathbf{n}(\tau)}{\partial\tau}\times\mathbf{n}(\tau) \tag{106}$$

Second, the second term in the Berry phase can be expressed as

$$a\sum_i\int_0^\beta d\tau[\frac{\partial\mathbf{n}(\tau)}{\partial\tau}\times\mathbf{n}(\tau)]\cdot\mathbf{L}_i\sim a\sum_i\int_0^\beta d\tau[\frac{\partial\mathbf{m}_i(\tau)}{\partial\tau}\times\mathbf{m}_i(\tau)]\cdot\mathbf{L}_i \tag{107}$$

Combining the Berry phase and the Hamiltonian term together, we reach

$$\begin{aligned}S_{eff} &= -iS\frac{1}{2}\int_0^\beta d\tau\int dx[\frac{\partial\mathbf{m}(x,\tau)}{\partial\tau}\times\mathbf{m}(x,\tau)]\cdot\frac{\partial\mathbf{m}(x,\tau)}{\partial x} \\ &+ \int_0^\beta d\tau\int dx[\frac{JS^2a^2}{2}|\nabla\mathbf{m}(x,\tau)|^2] + JS^2a^2\mathbf{L}(x,\tau)^2] - iSa[\frac{\partial\mathbf{m}_i(\tau)}{\partial\tau}\times\mathbf{m}_i(\tau)]\cdot\mathbf{L}_i \end{aligned} \tag{108}$$

We immediately notice that this action is quadratic in the variable $\mathbf{L}(x,\tau)$. We can then integrate out this variable and obtain the final result:

$$S_{eff} = -iS\frac{1}{2}\int_0^\beta d\tau \int dx[\frac{\partial \mathbf{m}(x,\tau)}{\partial \tau} \times \mathbf{m}(x,\tau)] \cdot \frac{\partial \mathbf{m}(x,\tau)}{\partial x}$$
$$+ \int_0^\beta d\tau \int dx[\frac{JS^2a^2}{2}\nabla_x^2 \mathbf{m}(x,\tau)] + \frac{1}{4Ja}(\frac{\partial \mathbf{m}(x,\tau)}{\partial \tau})^2 \quad (109)$$
$$\equiv \frac{1}{g}\int dx \int d\tau [v(\partial_x \mathbf{m}(x,\tau))^2 + \frac{1}{v}(\partial_\tau \mathbf{m}(x,\tau))^2] + \frac{i\theta}{8\pi}\epsilon_{ij}\mathbf{m}(x,\tau)\cdot(\partial_i \mathbf{m}(x,\tau)\times\partial_j \mathbf{m}(x,\tau)). \quad (110)$$

where $g = 2/S$, $v = 2aJS$ is the spin wave velocity and most importantly, $\theta = 2\pi S$. This is so-called non-linear σ model. (The d-dimensional case without Berry phase has been studied before.)

Since in each point of the space time $\mathbf{m}$ can also be viewed as an element of S^2, the mapping $\mathbf{m}(x,\tau)$ corresponds to an embedding of the sphere into the sphere ($(x,\tau) \to \mathbf{m}(x,\tau)$). Such embeddings are classified by what is called the second homotopy group of the sphere $\pi_2(S^2) = Z$. To each embedding corresponds an integer (element of Z) given by the Pontryagin index:

$$Q \equiv \frac{1}{8\pi}\int dx \int d\tau \epsilon_{ij}\mathbf{m}(x,\tau)\cdot(\partial_i \mathbf{m}(x,\tau)\times\partial_j \mathbf{m}(x,\tau)) \in Z \quad (111)$$

This number counts how often the unit sphere is wrapped by this projection. Q is called the Skyrmion number. We conclude that the first term on the right-hand side in action, the Berry phase, is given by $e^{-i2\pi SQ}$. In the case when S is an integer, because Q is also an integer, $e^{-i2\pi SQ} = 1$, and the Berry phase can be ignored. However, for a half-odd-integer spin, $e^{-i2\pi SQ} = (-1)^Q$, and depending on whether Q is odd or even, the sign does change.

That the low-energy properties of the 1D quantum spin system are drastically different between $S = 1, 2, 3, ...$ and $S = 1/2, 3/2, ...$ [Haldane, 1983]. Another general picture is, when spin S is an integer (where berry phase can be ignored), S_{eff} is always positive. In this case, quantum fluctuations in the path integral are not important. In contrast, for half-integer S, the quantum fluctuations is strong owing to quantum interference caused by the alternating berry phase. Therefore, the staggered component $\mathbf{m}(x,\tau)$ experiences less quantum effect for integer spin; on the other hand, it behaves stronger fluctuations in half-odd-integer spin case. As a result, the spin $\mathbf{m}(x,\tau)$ shows longer-range correlation in the half-odd integer spin. It is consistent with the gap values: For half-odd integer spin the

system is gapless (due to LSM), so the correlation length is $\xi \sim m^{-1} \to \infty$; for integer spin the system is gapped, with a finite correlation length.

Skyrmion quantum number.— The topological index of the field can be described mathematically as

$$Q \equiv \frac{1}{4\pi} \int dx \int d\tau \mathbf{m}(x,\tau) \cdot (\partial_x \mathbf{m}(x,\tau) \times \partial_y \mathbf{m}(x,\tau)) \tag{112}$$

where Q is the topological index, $\mathbf{m}$ is the unit vector in the direction of the local magnetization within the magnetic thin, ultra-thin or bulk film, and the integral is taken over a two dimensional space. (A generalization to a three-dimensional space is possible). Passing to spherical coordinates for the space $(x, y) = (r\cos\alpha, r\sin\alpha)$ and for the magnetisation $\mathbf{m} = (m\cos\phi\sin\theta, m\sin\phi\sin\theta, m\cos\theta)$, one can understand the meaning of the skyrmion number. Then the topological skyrmion number reads:

$$Q = \frac{1}{4\pi} \int \int \frac{d\theta}{dr}\frac{d\phi}{d\alpha} \sin\theta d\alpha dr = \frac{1}{4\pi} [\cos\theta]_{\theta(r=\infty)}^{\theta(r=0)} [\phi]_{\phi_f}^{\phi_i} = \pm W \tag{113}$$

where $+(-)$ is for the origin $\theta(r=0) = \pi(0)$ and W is the winding number. The winding number allows to define the skyrmion ($\phi(\alpha) \propto \alpha$ with a positive winding number and the antiskyrmion $\phi(\alpha) \propto -\alpha$ with a negative winding number.

The configuration is illustrated schematically (for a two dimensional skyrmion) in figure 4.

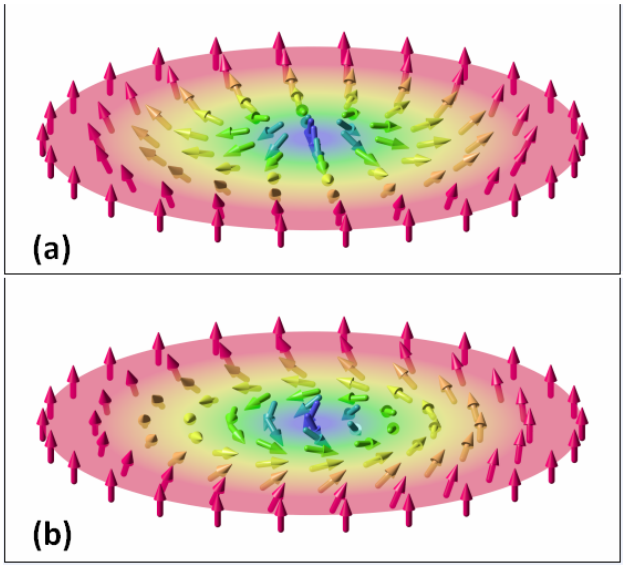


FIG. 4: The vector field of two-dimensional magnetic skyrmions: a) a hedgehog skyrmion and b) a spiral skyrmion.

2D ISING MODEL

The Ising model was originally introduced by Lenz in 1920, to describe the transition from a para- to a ferromagnetic phase in a magnetic lattice. The solution was given by Ising (1925) in dimensions $d = 1$. Later, it has become a paradigm for several different systems, including binary alloy, lattice gases, and large biological molecules (Huang's textbook). The reason of its relevance and popularity resides on the fact that it accounts for an order-disorder transition on a lattice by dealing with a minimum number of variables and external parameters. Beyond $d = 1$, the exact solution are available in two dimension $d = 2$, first in a vanishing external field [Onsager Phys. Rev. 65, 117 (1944); Kaufman Phys. Rev. 76, 1232 (1949)], and then in a nonzero external field (Yang, 1952). This allows to extract all details of the model, including its critical exponents, which can then be compared with approximate or numerical estimates of similar models. Actually, in quite a long time, it is the only non-trivial example of a phase transition that can be worked out with mathematical rigor. In three and more dimensions, a mean-field approximation is still capable of grasping most of the features of the Ising model.

Consider a d-dimensional lattice with N sites, and assume that the state of each lattice site, labeled by i, with $i = 1, ..., N$, can be characterized by the value of a single variable, say σ_i, taking only the possible values $\sigma_i = \pm 1$. For the sake of definiteness, we might think of magnetic spins residing on such sites, with $\sigma_i = 1$ corresponding to a spin up, and $\sigma_i = -1$ corresponding to a spin down. The Ising model is then a minimal model allowing for interaction between spins residing at nearest-neighbour sites in the lattice, and for spins with an external magnetic field, B say. The 'Hamiltonian' (i.e. the classical energy) of the model is then given by

$$H = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j - B \sum_i \sigma_i \tag{114}$$

Here, $-J$ is the interaction energy between sites i and j, and the summation restricts that to nearest-neighbouring sites only. In the following, for the sake of simplicity, we shall assume $-J < 0$. The number of nearest neighbours, or coordination number, z, is determined by the geometry of the lattice, being $z = 2d$ for a cubic lattice in d-dimensions.

The Ising model differs from the Heisenberg model in that the spins are purely classical.

They do not obey quantum commutation relations as do the spins in the Heisenberg model.

The partition function (in canonical ensemble) of the Ising model can be written as

$$Z = \sum_{\sigma_i=\pm,\ldots,\sigma_N=\pm} e^{-\beta H} \tag{115}$$

where the summation is over all possible values of spins. According to the laws of statistical mechanics, the partition function determines thermodynamic properties. For example, the thermal dynamic functions are obtained in the usual manner from the free energy:

$$F = -\beta \ln Z \tag{116}$$

Some other quantities such as the specific heat can be obtained by

$$C = \frac{\partial U}{\partial T}, U = -kT^2 \frac{\partial F}{\partial T} \tag{117}$$

The thermal average of the magnetization $M = \sum_i \sigma_i$ can be extracted from the partition function as

$$\langle M \rangle = \langle \sum_i \sigma_i \rangle = k_B T \frac{\partial}{\partial B} \log Z \tag{118}$$

The central interest about Ising model is the phase transition from an ordered state to a disordered state (details will be discussed below). Above the critical temperature T_c the system is in a disordered state, which corresponds to a random distribution of the spin values. Below the critical temperature T_c (nearly) all spins are aligned, even in the absence of an external applied magnetic field H. If we heat up a cooled ferromagnet, the magnetization vanishes at T_c and the ferromagnet switches from an ordered to a disordered state. This is a phase transition of second order. This will be the main topic in this course.

Mapping from two-dimension classical Ising model to quantum transverse Ising model

Let us move to study the case in 2d. The result will be different from the case in 1d.

We start with the partition function again

$$Z = \sum_{\sigma_1=\pm,\ldots,\sigma_N=\pm} \exp[-\beta H], H = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j \tag{119}$$

Since we are dealing with the model in two dimension (squared mesh), it is better to introduce coordinates (p, q) for each site, where $p, q \in Z$, and denote the coordinates in x- and y-direction, respectively. We generalize our model allowing for different couplings along the x- and y-directions, and having N sites in the x-direction and M-sites in the y-direction but keeping periodic boundary conditions (p.b.c) along both directions. With all these changes, the Hamiltonian can be written as follows:

$$H = \sum_{q=1}^{M} L(q, q+1) \tag{120}$$

$$L(q, q+1) = \sum_{p=1}^{N} (-J_x \sigma_{p,q}\sigma_{p+1,q} - J_y \sigma_{p,q}\sigma_{p,q+1}) \tag{121}$$

We consider first the case $J_x = 0$, that corresponds to N decoupled one-dimensional Ising models. Let us consider one of those chains, say the p-th one and at a given site q. Then, the partition function consists of a product of terms as follows,

$$Z_p = \sum_{\sigma_{p,1},\ldots,\sigma_{p,M}} \prod T^y_{p,q}, \quad T^y_{p,q} = e^{\beta J_y \sigma_{pq}\sigma_{p,q+1}} \tag{122}$$

Since the variables $\sigma_{p,q}$ have two possible values, we can represent them by a two component vector (a spinor):

$$\sigma_{p,q} = 1 = (1, 0)^T, \sigma_{p,q} = -1 = (0, 1)^T \tag{123}$$

such that the transfer matrix T^y_{pq} is the same as that in 1d case discussed in the previous section:

$$T^y_{\sigma_{pq},\sigma_{p,q+1}} = \begin{pmatrix} e^{\beta J_y} & e^{-\beta J_y} \\ e^{-\beta J_y} & e^{\beta J_y} \end{pmatrix}_{\sigma_{pq},\sigma_{p,q+1}} \tag{124}$$

Since a 2×2 matrix can be written in terms of Pauli matrices, we have

$$T^y = e^{\beta J_y} I + e^{-\beta J_y} \hat{\sigma}^x = e^{\beta J_y}(1 + e^{-2\beta J_y} \hat{\sigma}^x) \tag{125}$$

(Here transfer matrix T^y should be understood by $\langle \sigma_{p,q+1} | T^y | \sigma_{p,q} \rangle = e^{\beta J_y \sigma_{pq}\sigma_{p,q+1}}$.)

At this point we recall that T^y_{pq} is part of a partition function, and therefore, it would be easier to interpret what we have, if we could express it as the exponential of an operator. Since

$$e^{a\hat{\sigma}^x} = \cosh a + \sinh a \hat{\sigma}^x = \cosh a(1 + \tanh a \hat{\sigma}^x) \tag{126}$$

we can set $\tanh a = e^{-2\beta J_y}$, so we obtain

$$T^y = (\sinh a \cosh a)^{-1/2} \exp[a\hat{\sigma}^x] = (2\sinh(2J_y))^{1/2} \exp[a\hat{\sigma}^x] \tag{127}$$

where we used the relation

$$\tanh a = e^{-2J_y}, \cosh^2 a - \sinh^2 a = 1 \rightarrow \cosh^2 a = \frac{1}{1 - e^{-4J_y}} \tag{128}$$

Until now, we were discussing the one-dimensional Ising model. The corresponding partition function is

$$Z_p = \sum_{\sigma_{p,1},\ldots,\sigma_{p,M}} T^y_{\sigma_{p,1},\sigma_{p,2}} T^y_{\sigma_{p,2},\sigma_{p,3}} \ldots T^y_{\sigma_{p,M},\sigma_{p,1}} = Tr[(T^y)^M] \tag{129}$$

where T^y is the transfer matrix along y-direction. Since the trace is invariant under a unitary transformation, it is more informative to look at the trace after diagonalizing T^y, similar to the discussion in one-dimension case.

For the two-dimensional case, we have still to switch on J_x. Please note that, in above we have introduced matrix $\hat{\sigma}_x$ and defined the basis of $|\sigma_{p,q} = \pm 1\rangle$. Next we can consider two columns p and $p+1$, and we require the transfer matrix T^x satisfying

$$\langle \sigma_{p,q} | T^x | \sigma_{p+1,q} \rangle = e^{\beta J_x \sigma_{p,q} \sigma_{p+1,q}} \tag{130}$$

Here we see that the operator T^x should be such that its matrix elements contain no information on the states at $q+1$. One notices that the following form meets the requirements:

$$T^x_{pq} = \exp[\beta J_x \hat{\sigma}^z_{pq} \hat{\sigma}^z_{p+1,q}] \tag{131}$$

With the results above we arrive at the partition function for the whole system

$$Z = (2\sinh 2J_y)^{NM/2} Tr[T^M] \tag{132}$$

$$T = \exp[J_x \sum_p \hat{\sigma}^z_p \hat{\sigma}^z_{p+1}] \exp[a \sum_p \hat{\sigma}^x_p] \tag{133}$$

This is now the transfer matrix for the two-dimensional anisotropic Ising model. In contrast to the one-dimensional case, we have now instead of a 2×2 matrix, a $2^N \times 2^N$ dimensional array. It is however possible to solve the problem exactly, by means of a Jordan-Wigner transformation making fermions out of spins.

We first notice that the two exponentials $(\hat{\sigma}^x, \hat{\sigma}^z)$ do not commute with each other. This makes in fact the problem really quantum mechanical. Performing these replacements, we can write

$$Z \sim Tre^{-\beta H}, H = -J\sum_p \sigma_p^z \sigma_{p+1}^z - h\sum_p \sigma_p^x \tag{134}$$

This is the one-dimensional Hamiltonian of the Ising model with transverse field, that due to the presence of two noncommuting pieces is a genuinely quantum mechanical model. Please note that, no thermal fluctuations in the new model, but the quantum fluctuations appear. In this regarding, the above mapping provides an example the equivalence between $d+1$-dimensional classical model and $d-$dimensional quantum model.

The so-called transverse Ising model is a canonical model in the study of quantum critical point or quantum phase transition. It is the simplest model with a quantum critical point. It is exactly solvable, as we will show below. It also connects with the conformal field theory, so it is really very important.

Phase diagram

In the limit of $h \to 0$, the ground state is an Ising ferromagnet that spontaneously breaks Z_2 symmetry. The two-fold degenerate ground states are

$$|\Psi_\downarrow\rangle = |\downarrow_1\downarrow_2\cdots\rangle, |\Psi_\uparrow\rangle = |\uparrow_1\uparrow_2\cdots\rangle. \tag{135}$$

When $h \to \infty$, the ground state is a trivial paramagnet that preserves Ising symmetry,

$$|\Psi_x\rangle = |+_1+_2\cdots\rangle, + = \uparrow - \downarrow . \tag{136}$$

The above analysis matches the Monte Carlo simulations as shown in Fig. **??**. The global phase diagram of the model is easy to image. The interesting problem is, where is the transition point?

Duality

There is a duality transformation which defines new Pauli operators in a dual lattice

$$\tau_i^x = \sigma_i^z \sigma_{i+1}^z, \tau_i^z = \prod_{j\le i} \sigma_i^x \tag{137}$$

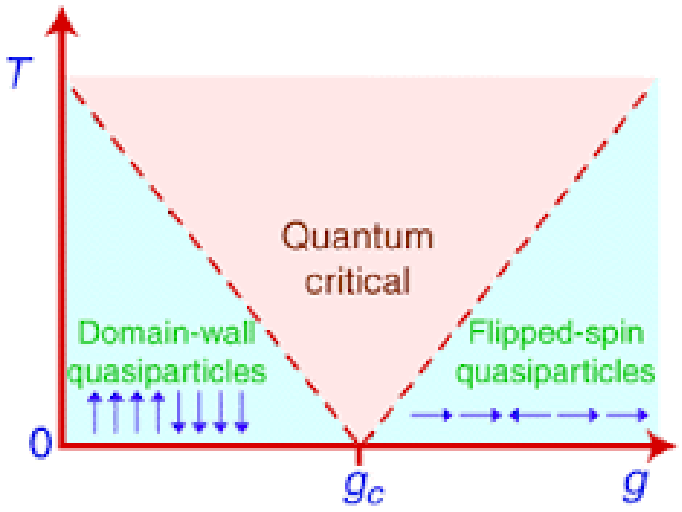


FIG. 5: Phase diagram of 1+1D transverse Ising model.

then these τ_i^x and τ_i^z satisfy the same commutation and anti-commutation relations of σ_i^x and σ_i^z, i.e. $\{\tau_i^a, \tau_j^b\} = 2\delta^{ab}$. And the original Hamiltonian can be written in terms of $\tau^{x,z}$ as

$$H = -J\sum_p \tau_p^x - h\sum_p \tau_{p+1}^z \tau_p^z \tag{138}$$

where we used the condition that $\tau_{p-1}^z \tau_p^z = \prod_{j<p} \sigma_i^x \prod_{i<p+1} \sigma_i^x = \sigma_p^x$

Since these two Hamiltonian take the same algebra, they should be the same (which means energy spectra, eigenvalues are all the same). But, we notice that the parameter exchange: $J \leftrightarrow h$. This is called duality. In this case, if there is a phase transition point, it should satisfy the condition of (by setting $J = 1$)

$$h_c = \frac{1}{h_c} \Rightarrow h_c = 1 \tag{139}$$

$\diamond$ *Homework:* Please prove the commutation relation between τ_i^x and τ_i^z.

The key feature is, we got a phase transition point $h_c \neq 0$ in 2d, which is quite different from that of 1d. The global phase diagram can be mapped out easily (see Fig. 5).

Wigner-Jordan transformation

Next we consider the 1+1 D transverse Ising chain with periodic boundary condition. The stategy is to use the fermionic representative [Two-dimensional Ising model as a soluble

problem of many fermions, T. D. Schultz, D. C. Mattis, E. H. Lieb]. Here we make the Jordan-Wigner transformation

$$c_n = \frac{\sigma_n^x + i\sigma_n^y}{2}\prod_{m<n}\sigma_m^z, c_n^\dagger = \frac{\sigma_n^x - i\sigma_n^y}{2}\prod_{m<n}\sigma_m^z, \tag{140}$$

$$\sigma_n^+ = \prod_{m<n}(1-2c_m^\dagger c_m)c_n, \sigma_n^- = \prod_{m<n}(1-2c_m^\dagger c_m)c_n^\dagger, \sigma_n^z = 1-2c_n^\dagger c_n \tag{141}$$

The string $\prod_{m<n}(1-2c_m^\dagger c_m)$ takes values ± 1, depending on even/odd number of fermions on the left side of n. One can check that,

$$\{c_n, c_m^\dagger\} = \delta_{m,n}, \{c_n, c_m\} = \{c_n^\dagger, c_m^\dagger\} = 0 \tag{142}$$

$$[\sigma_n^+, \sigma_m^-] = \delta_{n,m}\sigma_n^z, [\sigma_n^z, \sigma_m^\pm] = \pm 2\delta_{n,m}\sigma_n^\pm \tag{143}$$

(Only the Pauli matrix with the same site index should consider the commutation relation $\{\sigma_i^a, \sigma_j^b\} = 2\delta_{ij}\delta_{ab}, [\sigma_i^+, \sigma_j^-] = \delta_{ij}\sigma_j^z$.)

Under the Jordan-Wigner transformation, the Hamiltonian becomes

$$\begin{aligned} H &= \sum_n \sigma_n^z - \sum_n \sigma_n^x\sigma_{n+1}^x \\ &= \sum_{n=1}^{N}(1-2c_n^\dagger c_n) - \sum_{n=1}^{N-1}[c_n^\dagger c_{n+1}^\dagger + c_n^\dagger c_{n+1} + h.c.] + (c_N^\dagger c_1^\dagger + c_N^\dagger c_1 + h.c.)e^{i\pi\mathcal{N}}, \mathcal{N} = \sum_n c_n^\dagger c_n \end{aligned} \tag{144}$$

with

$$\begin{aligned} \sigma_n^x\sigma_{n+1}^x &= [\prod_{m<n}(1-2c_m^\dagger c_m)](c_m + c_m^\dagger)[\prod_{k<n+1}(1-2c_k^\dagger c_k)](c_k + c_k^\dagger) \\ &= (c_n^\dagger + c_n)(1-2c_n^\dagger c_n)(c_{n+1} + c_{n+1}^\dagger) \\ &= c_n^\dagger c_{n+1} + c_n^\dagger c_{n+1}^\dagger + h.c. \end{aligned} \tag{145}$$

The boundary term comes from that $\sigma_N^x\sigma_1^x = e^{i\pi\sum_{j<L}n_j}c_N^\dagger c_1 = -e^{i\pi\sum_{j\leq L}n_j}c_N^\dagger c_l = -e^{i\pi\mathcal{N}}c_N^\dagger c_l$, because to the left of $c_N^\dagger$ we certainly have $n_N = 1$. This shows that boundary condition are changed by fermion parity $e^{i\pi\mathcal{N}} = (-1)^{\mathcal{N}}$ and periodic boundary condition become anti-periodic boundary condition when $\mathcal{N}$ is even. And odd $\mathcal{N}$ relates to periodic boundary condition. Therefore, the real spin problem is not exactly the same with free fermion. Next, for odd $\mathcal{N}$, we set $e^{ikN} = 1, k = \frac{2\pi n}{N}, n = -N/2+1, ..., 0, ..., N/2$, for even $\mathcal{N}$, we set $e^{ikN} = -1, k = \pm\frac{\pi(2n-1)}{N}, n = 1, ..., N/2$.

In terms of momentum space $c_j = \frac{1}{\sqrt{N}} \sum_k e^{ikj} c_k$, the Hamiltonian becomes

$$\begin{aligned} H &= -\sum_k [2\cos(k) c_k^\dagger c_k + (e^{ik} c_k^\dagger c_{-k}^\dagger + h.c.)] + \sum_k (2c_k^\dagger c_k - 1) \\ &= \sum_k [(1-\cos(k))(c_k^\dagger c_k - c_{-k} c_{-k}^\dagger) - (e^{ik} c_k^\dagger c_{-k}^\dagger + h.c.)] \\ &= \sum_{k>0} [(1-\cos(k))(c_k^\dagger c_k - c_{-k} c_{-k}^\dagger) - (e^{ik} c_k^\dagger c_{-k}^\dagger + h.c.)] + \sum_{k<0} ... \\ &= \sum_{k>0} [2(1-\cos(k))(c_k^\dagger c_k - c_{-k} c_{-k}^\dagger) - (2i\sin(k) c_k^\dagger c_{-k}^\dagger - 2i\sin(k) c_{-k} c_k)] \\ &= \sum_{k>0} (c_k^\dagger, c_{-k}) \begin{pmatrix} 2(1-\cos(k)) & -2i\sin(k) \\ 2i\sin(k) & 2(1-\cos(k)) \end{pmatrix} \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix} \end{aligned} \tag{146}$$

where we used $\sum_k 2\cos(k) c_k^\dagger c_k = \sum_k \cos(k)(c_k^\dagger c_k - c_{-k} c_{-k}^\dagger)$, and $\sum_k (2c_k^\dagger c_k - 1) = \sum_k (c_k^\dagger c_k - c_{-k} c_{-k}^\dagger)$.

The diagonalization is akin to Bogovliubov transformation, and all eigenvalues can be calculated:

$$\Lambda(k) = \pm 2\sqrt{(\cos(k)-1)^2 + \sin^2(k)} = \pm 2\sin\frac{k}{2} \tag{147}$$

So we obtain that, for odd N, we have $e^{ikN} = 1, k = \frac{2\pi n}{N}, n = -N/2, ..., 0, ..., N/2$,

$$H = \sum_{n=0}^{N-1} \Lambda^-(n)(\eta_n^\dagger \eta_n - \frac{1}{2}) + const. \tag{148}$$

$$\Lambda^-(n) = [(1-\cos\frac{2\pi n}{N})^2 + (\sin\frac{2\pi n}{N})^2]^{1/2} = 2\sin\frac{2\pi n}{2N} \tag{149}$$

where Bogoliubov particle as $\begin{pmatrix} c_q \\ c_{-q}^\dagger \end{pmatrix} = \begin{pmatrix} u_q & -iv_q \\ -iv_q & u_q \end{pmatrix} \begin{pmatrix} \eta_q \\ \eta_{-q}^\dagger \end{pmatrix}$. (We have used that $H = \sum_{k>0} \Lambda(k)(\eta_k^\dagger \eta_k + \eta_{-k} \eta_{-k}^\dagger) = \sum_k \Lambda(k)(\eta_k^\dagger \eta_k - 1/2)$) For even N, we know the boundary condition is $e^{ikN} = -1, k = \pm\frac{\pi(2n-1)}{N}, n = 1, ..., N/2$.

$$H = \sum_{n=1}^{N/2} \Lambda^+(n)(\eta_n^\dagger \eta_n - \frac{1}{2}) + const. \tag{150}$$

$$\Lambda^+(n) = [(1-\cos\frac{\pi(2n-1)}{N})^2 + (\sin\frac{\pi(2n-1)}{N})^2]^{1/2} = 2\sin\frac{\pi(2n-1)}{2N} \tag{151}$$

The expression for H in above allows to immediately conclude that the ground state of the Hamiltonian must be the Bogoliubov vacuum state $|0\rangle$ which annihilates the $\vec{\eta}_k|0\rangle = 0$

for all k. Thus, the ground state energy is

$$E_0^+ = -\frac{1}{2}\sum_{n=1}^{N/2}\Lambda^+(n) + const. = -csc\frac{\pi}{2N} + const \approx -\frac{2N}{\pi} - \frac{\pi}{12N} + ... \quad (152)$$

$$E_0^- = -\frac{1}{2}\sum_{n=1}^{N/2-1}\Lambda^-(n) + const. = -cot\frac{\pi}{2N} + const \approx -\frac{2N}{\pi} + \frac{\pi}{6N} + ... = E_0^+ + \frac{\pi}{4N} \quad (153)$$

Compare E_0^+ ($1/N$ term) with CFT, we have $c = 1/2$. And we used $csc(x) \approx= \frac{1}{x} + \frac{x}{6} + .., cot(x) \approx \frac{1}{x} - \frac{x}{3} + ..$

The lowest excited energy in even sector is

$$E_1^+ = \Lambda^+(1) + \Lambda^+(N/2) + E_0^+ = 4\sin\frac{\pi}{2N} + E_0^+ \approx \frac{2\pi}{N} + E_0^+ \quad (154)$$

Thus, compared with CFT, we have

$$\Delta = \bar{\Delta} = 1/2 \quad (155)$$

In the odd sector, we have

$$E_0^- = E_0^+ + \frac{\pi}{4N} \quad (156)$$

Compared with CFT,

$$\Delta_\sigma = \bar{\Delta}_\sigma = 1/16 \quad (157)$$

This is related to the Majorana mode!

2D XY MODEL

Let us consider the XY (or rotator) model in a finite rectangle with periodic boundary conditions. (Actually, this is a preparation for future study. We will come back to this model again in future.) We include an external field in the z direction:

$$H = -\sum_{\langle i,j\rangle} \cos(\theta_i - \theta_j) - h \sum_j \cos\theta_j \tag{158}$$

Let N denote the number of sites in the rectangle and define

$$m = \frac{1}{N} \sum_j \langle \cos\theta_j \rangle \tag{159}$$

Then, Mermin-Wegner theorem states

$$\lim_{h\to 0} m = 0 \tag{160}$$

There are two ways to calculate it. One is to prove it using a phenomenogical method, and the other one is a direct calculation.

Low-temperature expansion

We calculate the magnetization as shown below.

In a low-temperature expansion, the angle difference between two spins will be small: $|\theta_i - \theta_j| \ll 2\pi$. In this small fluctuation regime, we can approximate the cosine term in the hamiltonian to extract the long-range behavior.

$$\begin{aligned} H &= -J \sum_{<i,j>} \cos(\theta_i - \theta_j) \\ &= -JN + \frac{J}{2} \sum_{<i,j>} (\theta_i - \theta_j)^2 \\ &= E_0 + \frac{J}{4} \sum_{\mathbf{r},\mathbf{a}} (\theta(\mathbf{r}+\mathbf{a}) - \theta(\mathbf{r}))^2 \\ &\simeq E_0 + \frac{J}{2} \int d^2 r (\nabla\theta(\mathbf{r})^2 \ . \end{aligned} \tag{161}$$

In the last line, we have taken the continuum limit, and replaced the field θ_i by a continuous one, $\theta(\mathbf{r})$, as slowly varying function of $\mathbf{r}$. From this, we can extract a lot of information about the magnetization and correlation functions.

Average magnetization

We calculate the average magnetization in the x direction for the 2D XY model (y is identical). We have:

$$\langle S_x \rangle = < \cos\theta(\mathbf{r}) > = < \cos\theta(0) > \tag{162}$$

$$= \frac{Tr_{\{\theta_i\}} \cos\theta(0) e^{-\beta H}}{Tr_{\{\theta_i\}} e^{-\beta H}} \tag{163}$$

$$= Re\left(\frac{1}{\mathcal{Z}} \int \mathcal{D}[\theta_i] e^{-\beta H + i\theta(0)}\right) \tag{164}$$

where $\mathcal{Z}$ is the partition function $Tr_{\{\theta_i\}} e^{-\beta H}$, and in the first line, we took advantage of translation invariance to set the spin at site $\mathbf{r} = 0$. In order to calculate that expression, we Fourier transform the θ variable, with periodic boundary conditions. We then have

$$\theta(\mathbf{r}) = \frac{1}{\sqrt{N}} \sum_{\mathbf{k}} \theta_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} \ , \tag{165}$$

$$\theta(\mathbf{r} = 0) = \frac{1}{\sqrt{N}} \sum_{\mathbf{k}} \theta_{\mathbf{k}} \tag{166}$$

This leads to, after Gaussian integral:

$$< S_x > = \exp\left(-\frac{T}{2J} I_d(L)\right) , \tag{167}$$

with $I_d(L)$ a geometric factor written as (by setting a momentum UV cut-off $\Lambda \sim \pi/a$)

$$I_d(L) = S_d \int_{\pi/L}^{\pi/a} dk k^{d-3} = \begin{cases} L^{2-d}, d < 2 \\ \ln\left(\frac{L}{a}\right), d = 2 \\ \frac{1}{d-2}\left(\frac{\pi}{a}\right)^{d-2}, d > 2 \end{cases} \tag{168}$$

Therefore,

$$\lim_{L\to\infty} \langle S_x \rangle = \begin{cases} 0, d \leq 2 \\ \exp(-\frac{S_d}{2Ja^{2-d}} AT), d > 2 \end{cases} \tag{169}$$

Then, for any $T \neq 0$ and $d = 2$, the logarithmic divergence of this geometric factor will force $< S_x >= 0$. This is directly the statement of the Mermin-Wagner theorem. Hence there can be no ordered low-temperature phase (in the conventional long-range order) in the 2D XY model.

Correlation functions

We now set on the same path, but for the spin-spin correlation function in $d = 2$:

$$g(r) =< \exp\{i(\theta(\mathbf{r}) - \theta(0))\} >=< \frac{Tr_{\{\theta_i\}} e^{i(\theta(\mathbf{r})-\theta(0))} e^{-\beta H}}{Tr_{\{\theta_i\}} e^{-\beta H}} >$$
$$= e^{-\frac{1}{2}<[\theta(\mathbf{r})-\theta(0)]^2>} = e^{-<\theta^2(0)-\theta(\mathbf{r})\theta(0)>}. \tag{170}$$

This is a conclusion for a Gaussian Hamiltonian: $< \exp\{i(\theta(\mathbf{r}) - \theta(0))\} >= e^{-\frac{1}{2}<[\theta(\mathbf{r})-\theta(0)]^2>}$. Then we have

$$\begin{aligned} < \theta^2(0) - \theta(\mathbf{r})\theta(0) > &= \int \frac{d^2k_1}{(2\pi)^2} \int \frac{d^2k_2}{(2\pi)^2} < \theta(k_1)\theta(k_2) > (1 - e^{i\mathbf{k_1}\cdot\mathbf{r}}) \\ &= T \int_0^\Lambda \frac{1 - e^{i\mathbf{k_1}\cdot\mathbf{r}}}{k^2} \\ &= T\frac{1}{4\pi^2} \int_0^{2\pi} d\phi \int_0^\Lambda dk \frac{1}{k}(1 - e^{ikr\cos\phi}) = T\frac{1}{2\pi} \int_0^\Lambda dk \frac{1 - J_0(kr)}{k} \\ &\approx \frac{T}{2\pi} \ln \frac{r}{a} \end{aligned} \tag{171}$$

So

$$g(r) = e^{-\frac{T}{2\pi}\ln\frac{r}{a}} = (\frac{r}{a})^{-\eta(T)}, \eta = \frac{T}{2\pi} \tag{172}$$

and we can conclude from this that, at low-temperatures, the XY model has an quasi-long-range correlations with index η temperature dependent. This means that, at all low-temperatures, the system is critical.

◇ Think about is: *How can there be a finite temperature transition in two-dimension?*

Vortices and entropic argument

Vortices are topological defects of the field $\theta(\mathbf{r})$, satisfying the Laplace equation $\nabla^2\theta(\mathbf{r}) = 0$. Apart from the trivial solution to this equation ($\theta(\mathbf{r}) = 0$, the ferromagnetic ground state), there are solutions called vortices. For a single vortex situated at $\mathbf{r}_0$, the circulation loop integral around it needs to be quantized:

$$\oint_{\mathbf{r}_0} \nabla\theta(\mathbf{r}) \cdot d\mathbf{l} = 2\pi n, \tag{173}$$

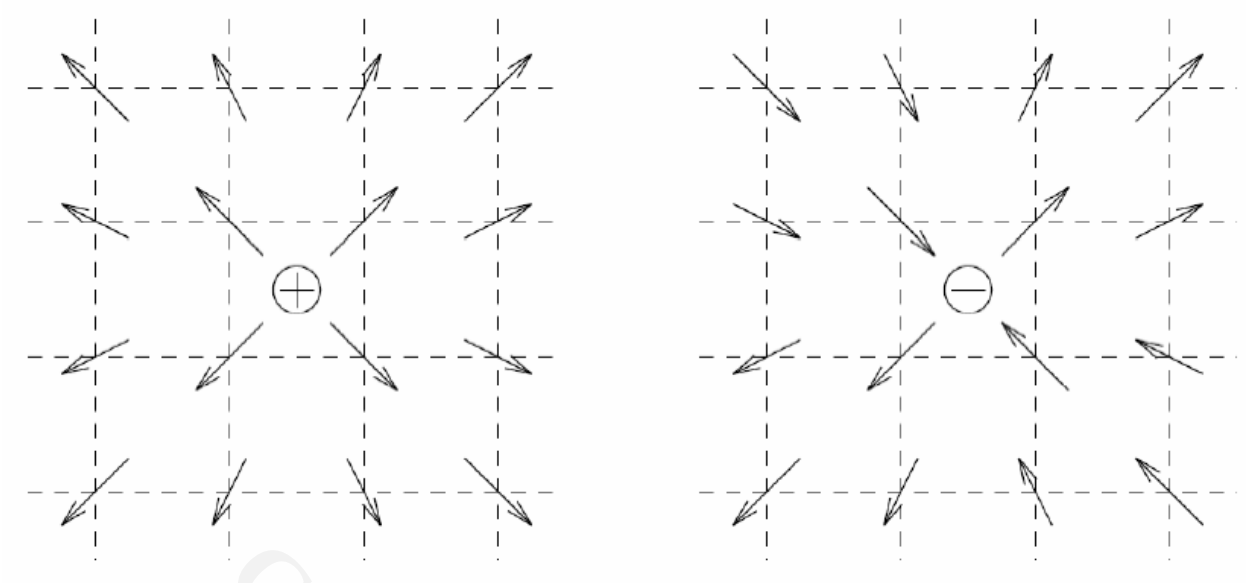


FIG. 6: (a) Following the vectors around the plaquette in a counterclockwise way, the vectors turn 2π while we circle 2π. This is a vortex. Its core radius is r, therefore the energy of such a vortex is $E \sim \ln(R/r)$. In (b), the vectors wind by -2π while we circle counterclockwise - this is an antivortex.

with $n < 0$ corresponding to clockwise winding vortices, and $n > 0$ to anticlockwise. Such configurations are illustrated in Fig. 6.

Can the proliferation of these objects be a reason for the quasi-long-range order? To estimate this, we consider the cost to the free energy $\Delta F = \Delta E - T\Delta S$ of adding a free vortex into a system without vortex. In order to estimate the energy generated by the presence of an isolated vortex, we must first estimate $\nabla\theta$. We use our equation 173, from which we estimate that, if there is one vortex on the lattice, then $\nabla\theta = \frac{n}{r}\hat{\theta}$. Therefore, the energy difference associated with this isolated vortex is

$$\Delta E = \frac{J}{2}\int d^2r(\nabla\theta(\mathbf{r}))^2 = \pi J n^2 \int_a^L \frac{dr}{r} = \pi J n^2 \ln\frac{L}{a}\,, \tag{174}$$

with L being the linear dimension of the system. We note that is a truly continuous system, we would have to start the integral at 0. However, our integral would then be divergent. It is therefore important here to consider the fact that all of this truly takes place on a lattice, where we have a lower spatial bound to this integral, the lattice constant a.

We then calculate the entropic cost to the creation of a vortex. We have that $\Delta S =$

$k_B \ln \Omega$, with Ω being the number of microstates that can be occupied with one vortex. Since we work on a lattice of size L^2 with a lattice constant a, this means there are $(L/a)^2$ ways to put this one vortex on the lattice. Hence, we have:

$$\Delta S = k_B \ln (L/a)^2 = 2k_B \ln L/a \ . \tag{175}$$

Hence, the cost in free energy to the creation of an isolated vortex is, in this heuristic approximation,

$$\Delta F = \Delta E - T\Delta S = (\pi J n^2 - 2k_B T) \ln \frac{L}{a} \ . \tag{176}$$

We can clearly see the following two regimes:

- For $k_B T < \pi J/2$, $\Delta F > 0$, and then isolated vortices are unfavourable. If they exist at all in the system, it will be in neutral pairs, where their effect at long distance is negated;
- For $k_B T > \pi J/2$, $\Delta F < 0$, and then isolated vortices are favourable and proliferate.

This provides us with our first crude estimate for the KT transition: $k_B T_c = \pi J/2$. We can now say that it is the unchecked proliferation of free vortices that kills the quasi-long-range order and leads to disorder. This remarkably simple argument from Kosterlitz and Thouless is not too far from the truth; one has to include the effect of the screening of ambient vortex pairs in the system to the interaction strength J to get a faithful and complete picture. To further probe this mechanism, we need to map the spin model to that of the 2D Coulomb gas and proceed with a renormalization group analysis.

Kosterlitz-Thouless Transition

Let $\mu = x, y$ label the unit vectors along the nearest-neighbor bonds on the square lattice. The partition function can be written as

$$Z = \int \prod_i \frac{d\theta_i}{2\pi} e^{\frac{J}{T}\sum_{i,\mu}\cos(\theta_i - \theta_{i+\mu})} = \int \prod_i \frac{d\theta_i}{2\pi} e^{\frac{J}{T}\sum_{i,\mu}\cos(\Delta_\mu \theta_i)} \tag{177}$$

Next we would like to introduce a Fourier transformation:

$$F(m_{i\mu}) = \frac{1}{2\pi}\int_0^{2\pi} d(\Delta_\mu \theta_i) e^{-i m_{i\mu}\Delta_\mu \theta_i} e^{\frac{J}{T}\cos(\Delta_\mu \theta_i)} = I_{m_{i\mu}}(\frac{J}{T}) \tag{178}$$

where $I_\nu(x)$ is modified Bessel function. And we assume the temperature is low, $J/T \gg 1$, so we take asymphobic form: $I_\nu(x) \approx \frac{1}{\sqrt{2\pi x}} e^{-\frac{\nu^2}{2x}}$. Plugging in this form into partition function,

$$e^{\frac{J}{T}\cos(\Delta_\mu \theta_i)} \approx \sqrt{\frac{T}{2\pi J}} e^{\frac{J}{T}} \sum_{m_{i\mu}=-\infty}^{+\infty} e^{i m_{i\mu}\Delta_\mu \theta_i - \frac{T m_{i\mu}^2}{2J}} \tag{179}$$

$$\begin{aligned} Z &= \int_0^{2\pi} \prod_i \frac{d\theta_i}{2\pi} \sum_{m_{i\mu}} e^{-\frac{T}{2J}\sum_{i\mu} m_{i\mu}^2 + i\sum_{i\mu} m_{i\mu}\Delta_\mu \theta_i} \\ &= \int_0^{2\pi} \prod_i \frac{d\theta_i}{2\pi} \sum_{m_{i\mu}} e^{-\frac{T}{2J}\sum_{i\mu} m_{i\mu}^2 + i\sum_{i\mu} m_{i\mu}(\theta_{i+\mu} - \theta_i)} \\ &= \int_0^{2\pi} \prod_i \frac{d\theta_i}{2\pi} \sum_{m_{i\mu}} e^{-\frac{T}{2J}\sum_{i\mu} m_{i\mu}^2 + i\sum_{i\mu}(m_{i\mu} - m_{i-\mu,\mu})\theta_i} \\ &= \sum_{m_{i\mu}} \delta(\vec{\Delta}\cdot\vec{m}_i) e^{-\frac{T}{2J}\sum_{i\mu} m_{i\mu}^2} \end{aligned} \tag{180}$$

At the last line, we integral θ_i and give a constraint

$$\vec{\Delta}\cdot\vec{m}_i = \sum_\mu \Delta_\mu m_{i\mu} = \sum_\mu (m_{i+\mu,\mu} - m_{i\mu}) = 0 \Rightarrow \sum_\mu m_{i\mu} = 0 \tag{181}$$

which means the lattice divergence of $\vec{m}$ field of every site is zero.

Next we deal with this constraint. We introduce a dual lattice, where integer variables n defined as:

$$m_{i,x} = n_{i+\frac{x+y}{2}} - n_{i+\frac{x-y}{2}} \tag{182}$$

$$m_{i,y} = n_{i+\frac{y-x}{2}} - n_{i+\frac{x+y}{2}} \tag{183}$$

$$m_{i,-x} = n_{i-\frac{x+y}{2}} - n_{i+\frac{y-x}{2}} \tag{184}$$

$$m_{i,-y} = n_{i+\frac{x-y}{2}} - n_{i-\frac{x+y}{2}} \tag{185}$$

so that one can check the above constraint $\sum_\mu m_{i\mu} = 0$ is satisfied automatically.

So we get the partition function defined on the dual lattice

$$Z = \sum_{n_i} e^{-\frac{T}{2J}\sum_{i\mu}(n_{i+\mu} - n_i)^2} \tag{186}$$

$$= \int \prod_i d\phi_i \sum_{n_i} e^{-\frac{T}{2J}\sum_{i\mu}(\phi_{i+\mu} - \phi_i)^2 - 2\pi i \sum_i \phi_i n_i} \tag{187}$$

In the last line, we used the Possion formula:

$$\sum_{n=-\infty}^{\infty} g(n) = \sum_{n=-\infty}^{\infty} \int d\phi g(\phi) e^{-2\pi i n \phi} \tag{188}$$

At this step, if we integral out ϕ field, the partition function is like $\sum_{n_i,n_j} \exp[n_i n_j \ln |\mathbf{r}_i - \mathbf{r}_j|]$ (see the above section), which is like the interaction term of 2D Coulomb gas. This term will be divergent, when $x_i = x_j$. To remove this divergence, we introduce the "core" energy of each vortex E_c, which cancels the divergence when $x_i = x_j$. Adding this term back to the action, we obtain

$$Z = \int \prod_i d\phi_i \sum_{n_i} e^{-\frac{T}{2J}\sum_{i\mu}(\phi_{i+\mu}-\phi_i)^2 - 2\pi i \sum_i \phi_i n_i} e^{-\frac{E_c}{T}\sum_i n_i^2} \tag{189}$$

where $y = e^{-\frac{E_c}{T}}$ is also called fugacity of vortices.

Next we have to use an approximation as

$$\sum_{n_i} e^{-2\pi i n_i \phi_i} y^{n_i^2} \approx 1 + 2y\cos(2\pi\phi_i) + 2y^4\cos(4\pi\phi_i) + ... \approx e^{2y\cos(2\pi\phi_i)+O(y^2)} \tag{190}$$

and we re-write the partition function as

$$\int \prod_i d\phi_i e^{-\frac{T}{2J}\sum_{i\mu}(\phi_{i+\mu}-\phi_i)^2 + 2y\sum_i \cos(2\pi\phi_i)} \tag{191}$$

where ϕ_i is dual or disorder variable. ϕ_i are ordered at high T and disordered at low T (opposite to θ_i). In the continuum limit, it is

$$Z = \int D\phi e^{-\int d^2x [\frac{T}{2J}(\nabla\phi)^2 + 2y\cos(2\pi\phi)]} \tag{192}$$

This is the sin-Gordon model, which is dual to the XY model.

Renormalization-group analysis

We separate the fast and slow modes $\phi = \phi_< + \phi_>$, and $\phi_<$ contains only the Fourier components with $k < \Lambda/b$. The partition function becomes

$$Z = \int D\phi_< D\phi_> e^{-\int d^2x[\frac{T}{2J}(\nabla\phi_<)^2+(\nabla\phi_>)^2} e^{2y\int d^2x\cos(2\pi(\phi_<+\phi_>))]} \tag{193}$$

$$\approx \int D\phi_< D\phi_> e^{-\int d^2x[\frac{T}{2J}(\nabla\phi_<)^2+(\nabla\phi_>)^2} \times$$

$$[1 + 2y\int d^2x\cos(2\pi(\phi_< + \phi_>)) + 2y^2\int d^2x\int d^2x'\cos(2\pi(\phi_<(x) + \phi_>(x)))\cos(2\pi(\phi_<(x') + \phi_>(x')))]$$

$$= Z_< \int D\phi_> e^{-\int d^2x\frac{T}{2J}(\nabla\phi_>)^2}[1 + 2y\int d^2x\cos(2\pi\phi_<)\langle\cos(2\pi\phi_>)\rangle_> +$$

$$2y^2\int d^2x\int d^2x'\cos(2\pi\phi_<(x))\cos(2\pi\phi_<(x')))\langle\cos(2\pi\phi_>(x)\cos(2\pi\phi_>(x'))\rangle_> +$$

$$2y^2\int d^2x\int d^2x'\cos(2\pi\phi_<(x))\cos(2\pi\phi_<(x')))\langle\sin(2\pi\phi_>(x)\sin(2\pi\phi_>(x'))\rangle_>] \tag{194}$$

Defining th correlation function

$$g_>(\mathbf{r}) = (2\pi)^2\langle\phi_>(\mathbf{r})\phi_>(0)\rangle_> \tag{195}$$

and

$$\langle\cos(2\pi\phi_>(\mathbf{r}))\rangle = e^{-\frac{1}{2}g_>(0)} \tag{196}$$

$$\langle\cos(2\pi\phi_>(\mathbf{r}))\cos(2\pi\phi_>(\mathbf{r}'))\rangle = e^{-g_>(0)}\cosh(g_<(\mathbf{x}-\mathbf{x}')) \tag{197}$$

$$\langle\sin(2\pi\phi_>(\mathbf{r}))\sin(2\pi\phi_>(\mathbf{r}'))\rangle = e^{-g_>(0)}\sinh(g_<(\mathbf{x}-\mathbf{x}')) \tag{198}$$

The action for the slow modes is

$$S_< = \frac{T}{2J}\int d^2x(\nabla\phi_<(\mathbf{r}))^2 - 2ye^{-\frac{1}{2}g_>(0)}\int d^2x\cos(2\pi\phi(x))$$

$$- y^2e^{-g_>(0)}\int dxdy[\cos(2\pi(\phi_<(x) + \phi_<(y)))(e^{-g_>(x,y)} - 1) + \cos(2\pi(\phi_<(x) - \phi_<(y)))(e^{g_>(x,y)} - 1)] \tag{199}$$

To the first order in fugacity, the result of the momentum shell integration gives the change of fugacity:

$$y \to y(b) = b^2ye^{-\frac{1}{2}g_>(0)} \tag{200}$$

$$g_>(0) = \frac{2\pi}{T}\int_{\Lambda/b}^{\Lambda}\frac{dk}{k} = \frac{2\pi}{T}\ln b \tag{201}$$

The renormalization of temperature derives from the second-order terms in $S_<$. Expanding $\phi_<(y)$ around x, to the leading order in gradients one can write

$$\cos(2\pi(\phi_<(x)-\phi_<(x'))) \approx 1-\frac{1}{2}(2\pi(x'-x)\cdot\nabla\phi_<(x))^2 \tag{202}$$

$$\cos(2\pi(\phi_<(x)+\phi_<(x'))) \approx \cos(4\pi\phi_<(x)) - 2\pi\sin(4\pi\phi_<(x))((x'-x)\cdot\nabla\phi_<(x)) \tag{203}$$

Then we got

$$T\to T(b) = T + 2\pi^2 y^2 e^{-g_>(0)}\int d^2x(e^{g_>(x)}-1)x^2 \tag{204}$$

$$g_>(x) = \frac{1}{T}\int_{\Lambda/b}^{\Lambda}\frac{dk}{k}\int_0^{2\pi} d\alpha e^{ikx\cos(\alpha)} = \frac{2\pi}{T}\ln b J_0(\Lambda x) \tag{205}$$

Here we need treat this integral carefully.

$$\begin{aligned} g_>(x) &= \int_0^{\Lambda}\frac{d^2q e^{i\mathbf{q}\cdot\mathbf{x}}}{Tq^2} - \int_0^{\Lambda/b}\frac{d^2q e^{i\mathbf{q}\cdot\mathbf{x}}}{Tq^2} \\ &\approx \frac{2\pi}{T}\Big(\int_0^{\infty} dq\frac{qJ_0(qx)}{q^2+(\Lambda/b)^2} - \int_0^{\infty} dq\frac{qJ_0(qx)}{q^2+(\Lambda)^2}\Big) \\ &= \frac{2\pi}{T}(K_0(x\Lambda/b)-K_0(x\Lambda)) \qquad\qquad = -\frac{2\pi x\Lambda\ln b}{T}\frac{dK_0(z)}{dz}_{z=x\Lambda} \end{aligned} \tag{206}$$

$$\begin{aligned} T\to T(b) &= T - 2\pi^2 y^2\ln b\int d^2x x^2\frac{2\pi x\Lambda}{T}\frac{dK_0(z)}{dz}_{z=x\Lambda} \\ &= T + \frac{1}{2T}(\frac{y(4\pi)^2}{\Lambda^2})^2\ln b \end{aligned} \tag{207}$$

Define $y = \frac{y(4\pi)^2}{\Lambda^2}$, the flow of couplings in the sine-Gordon theory becomes

$$\begin{aligned} \frac{dT}{d\ln b} &= y^2/2T + O(y^4)\ , \\ \frac{dy}{dl} &= (2-\pi/T)y + O(y^3)\ , \end{aligned} \tag{208}$$

The critical point is $T_{KT} = \pi/2$. As shown in Figure, in low temperature regime $T < T_{KT}$, the fixed point line relates to $y^* = 0$. $y\to 0$ means vainishing probability to find vortices, or no free vortices in the system. All vortices form vortex-anti-vortices pairs. When $T > T_{KT}$, free vortices appear.

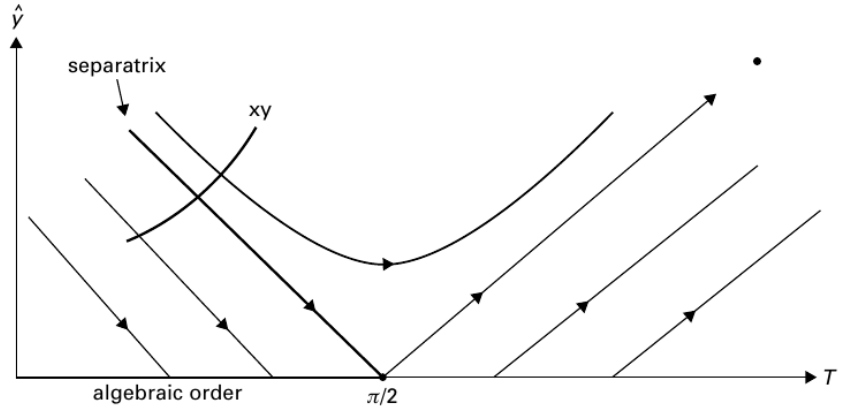


FIG. 7: Renormalization group flow of the temperature and the fugacity in the sine-Gordon model. The thick line separates the flows towards the line of fixed points at $T < T_{KT}$ and $y = 0$ that represent the algebraically ordered superfluid phase from the flow towards a high-temperature high-fugacity sink, representing the exponentially disordered phase. The XY line represents the set of initial values at different temperatures in the XY model.

ABC OF 2D CONFORMAL FIELD THEORY

Brief Introduction

We need to start from two ingredients. On one hand, quantum mechanism deals with the microscopic behavior of single particle. Quantum field theory describes the quantum physics of many-particles or interaction between particles. On the other hand, symmetry plays an important role on the physics. Examples include Lorentz symmetry for relativity, $U(1)$ gauge symmetry for Maxwell theory. Now we study a very special symmetry on QFT. This symmetry constrains the behavior of QFT, giving many direct answers without detailed calculations. The procedure looks very different from the formal QFT.

A conformal field theory (CFT) is a field theory that is invariant under conformal transformations. It means that the metric transforms with a scale factor, thereby preserving the angle between two vectors

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x) = \Lambda(x) g_{\mu\nu}(x). \tag{209}$$

The four main operations that have this property are translations, rotations, dilations and the so-called special conformal transformations. All these operations form a group, dubbed as conformal group. The CFT is a quantum field theory that is invariant under conformal group.

In physics, conformally invariant quantum field theories (are believed to) describe the critical behavior of systems at second order phase transitions. The canonical example is the Ising model in 1+1 dimension, at the phase transition, typical configurations have fluctuations on all length scales, so the field theory at its critical point should be expected to be invariant under changes of scale. In two dimensions, the conformal algebra becomes infinite dimensional, leading to significant restrictions on two dimensional conformally invariant theories, and perhaps ultimately giving a classification of possible critical phenomena in two dimensions. Rigidly speaking, the CFT in most of physical systems are emergent, since the conformal symmetry is not exact or explicitly exists. Usually, we only have scaling invariance at the critical point, and the other symmetries are argued to be "emergent".

It is first of all natural to write the coordinates in a complex notation: $z = x + iy$ and $\bar{z} = x - iy$, because the conformal mappings in two dimensions are then (anti-)holomorphic functions. The most important fields in a CFT are the primary fields, transforming under

conformal transformations as

$$\phi(z,\bar{z}) = \left(\frac{\partial w}{\partial z}\right)^{h} \left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^{\bar{h}} \phi(w,\bar{w}) \tag{210}$$

where $h, \bar{h}$ are the conformal weights of the primary field. Their sum, $h + \bar{h} = \Delta$, is called the scaling weight or dimension of the field. The transformation property above heavily constraints the expectation value of two of these primary fields, it has to have the form

$$\langle \phi(z_1)\phi(z_2)\rangle = \frac{c_{12}}{(z_1 - z_2)^{2h}}, \tag{211}$$

where c_{12} is a constant, $h = h_1 = h_2$ and only the chiral ($\bar{z}$-independent) part is considered. If the two fields in the correlator do not have the same conformal weight, the result will be zero. The three-point correlator is also heavily restricted,

$$\langle \phi(z_1)\phi(z_2)\phi(z_3)\rangle = \frac{c_{123}}{(z_1 - z_2)^{h_{123}}(z_2 - z_3)^{h_{231}}(z_3 - z_1)^{h_{312}}}, \tag{212}$$

Here, the short hand notations $h_{123} = h_1 + h_2 - h_3$ have been used. The form for higher order correlators is not completely fixed by the transformation rules.

An important concept in CFT is the operator product expansion (OPE). When two fields in a correlator are near each other, they will behave singular and can act as if the fields have fused together in one or more other fields. This information is encoded in the OPE of these two fields,

$$\lim_{z\to w} \phi_i(z)\phi_j(w) = \sum_k C_{ijk}\phi_k(z)(z-w)^{-\Delta_i - \Delta_j + \Delta_k} \tag{213}$$

where the sum is taken over all the primary fields in the theory. The constants C_{ijk} are called the structure constants. The two fields will thus fuse together to form one or more other fields times a singular coefficient that depends on the conformal dimensions of all three fields present in the fusion process. The fusion rules themselves can be written down more simply in the form

$$\phi_i \times \phi_j = N_{ijk}\phi_k \tag{214}$$

The fusion process is both associative and commutative, and results in an algebra of the constants N_{ijk}, called the Verlinde algebra. If the OPE of two fields contains multiple nonzero structure constants, there will be multiple fusion channels and the fusion is called nontrivial. These nontrivial fusion rules are important, because fields with nontrivial fusion rules correspond to non-abelian particles. This property can make calculating the correlator

of multiple fields hard, because there can be many different ways in which all the fields fuse together. A necessary condition for a correlator to be nonzero is that there has to be at least one fusion path resulting in the identity field, because only the expectation value of the identity field is nonzero. If, for example, the fusion rule for some fields is $\eta \times \eta = \xi$, then $\langle \xi \rangle = 0$, because these two fields do not fuse to the identity. In general, the correlator is a sum of all the different ways in which the fields can be fused together,

$$\langle \phi(z_1)...\phi(z_N) \rangle = F(z_1, ..., z_N). \tag{215}$$

Here, the F are all the possible resulting (holomorphic) functions and they are called conformal blocks. A consequence is that a correlator is not always just equal to one holomorphic function, but can be seen as a vector in the space of the many possible resulting functions. It can be shown that displacing the z_i in the fields on the left hand side results in the conformal blocks transforming into each other, this is the braiding of the conformal blocks.

A special OPE is that of a primary field with the energy-momentum tensor T, given by

$$T(z)\phi(w) = \frac{h}{(z-w)^2}\phi(w) + \frac{1}{z-w}\partial\phi(w) + ... \tag{216}$$

The dots at the right hand side of this formula represent all nonsingular terms, they are usually omitted in the OPE, because they play no role in the calculation of correlators. This relation is very useful, and is often used to determine the conformal dimension h of a field.

A most useful primary fields are the vertex operators, e.g. $V(z) = e^{i\alpha\phi(z)}$ in fractional quantum Hall theories (e.g. Laughlin state). Here, the $\phi(z)$ are chiral bosonic fields, meaning that they have no $\bar{z}$ dependence. The correlator of two of those ϕs is $\langle \phi(z)\phi(w) \rangle = -\ln(z-w)$. To be continue...

Global conformal symmetry

d-dimensional conformal transformation

We denote by $g_{\mu\nu}(x)$ the metric tensor in a space-time of dimension d: $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. By definition, a conformal transformation of the coordinates is an invertible mapping, which leaves the metric tensor invariant up to a scale (compared with the general case $g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}g_{\alpha\beta}(x)$):

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \Lambda(x)g_{\mu\nu}(x). \tag{217}$$

A conformal transformation $w = f(z)$ preserves local angles. Generally speaking, a conformal transformation is a coordinate transformation that is a local rescaling of the metric, which is a subgroup of the whole coordinate transformations. The set of conformal transformations manifestly forms a group, and it obviously has the Poincare group as a subgroup (in flat space), since the latter corresponds to the special case $\Lambda(x) = 1$.

To study the properties of a given group, we can start from the infinitesimal generators. The infinitesimal generators of the conformal group can be determined by considering the infinitesimal coordinate transformation $x^\mu \to x^\mu + \epsilon^\mu(x)$, under which

$$\begin{aligned} & ds^2 = g_{\mu\nu}dx^\mu dx^\nu = g'_{\mu\nu}dx'^\mu dx'^\nu \\ \to\ & g_{\mu\nu}dx^\mu dx^\nu = (g_{\mu\nu} + \delta g_{\mu\nu})d(x^\mu + \epsilon^\mu(x))d(x^\nu + \epsilon^\nu(x)) \\ =\ & g_{\mu\nu}dx^\mu dx^\nu = g_{\mu\nu}dx^\mu dx^\nu + g_{\mu\nu}(d\epsilon^\mu dx^\nu + d\epsilon^\nu dx^\mu) + \delta g_{\mu\nu}dx^\mu dx^\nu \\ \to\ & \delta g_{\mu\nu}dx^\mu dx^\nu = -(\frac{\partial\epsilon_\mu}{\partial x^\nu} + \frac{\partial\epsilon_\nu}{\partial x^\mu})dx^\mu dx^\nu \end{aligned}$$

thus we have

$$g_{\mu\nu}(x) \to g_{\mu\nu}(x) - (\frac{\partial\epsilon_\mu}{\partial x^\nu} + \frac{\partial\epsilon_\nu}{\partial x^\mu}) \tag{218}$$

The requirement that the transformation be conformal implies that

$$(\frac{\partial\epsilon_\mu}{\partial x^\nu} + \frac{\partial\epsilon_\nu}{\partial x^\mu}) = f(x)g_{\mu\nu}(x) \tag{219}$$

where $f(x) = \frac{2}{d}\partial_\mu\epsilon^\mu = \frac{2}{d}(\partial\cdot\epsilon)$ is obtained by trace on two sides ($g^{\mu\nu}(\frac{\partial\epsilon_\mu}{\partial x^\nu} + \frac{\partial\epsilon_\nu}{\partial x^\mu}) = f(x)g^{\mu\nu}(x)g_{\mu\nu}(x) \Rightarrow 2\partial_\mu\epsilon^\mu = df(x)$).

Next, we apply derivative on $\partial^\rho\epsilon_\rho$ and have

$$\partial_\nu\partial_\mu(\partial^\rho\epsilon_\rho) = \partial_\mu\partial^\rho(\partial_\nu\epsilon_\rho + \partial_\rho\epsilon_\nu - \partial_\rho\epsilon_\nu) = \frac{2}{d}\partial_\nu\partial_\mu(\partial\cdot\epsilon) - \partial^\rho\partial_\rho\partial_\mu\epsilon_\nu \tag{220}$$

and

$$(g_{\mu\nu}\partial^\rho\partial_\rho + (d-2)\partial_\mu\partial_\nu)(\partial\cdot\epsilon) = 0 \tag{221}$$

Taking the trace, we then get

$$(d-1)\partial^\mu\partial_\mu(\partial\cdot\epsilon) = 0. \tag{222}$$

These two equation imply that, 1) if $d = 1$, the above equations do not impose any constraint on the function $\partial\cdot\epsilon$, and therefore any smooth transformation is conformal in one dimension.

This is a trivial statement, since the notion of angle then does not exist. 2) $d = 2$ is special, which is different from $d > 2$. 3) in $d > 2$, the condition $\partial_\mu \partial_\nu (\partial \cdot \epsilon) = 0$ enforces that $\partial \cdot \epsilon$ is a linear function like $\partial \cdot \epsilon = A + B_m x_m$. If we substitute this expression into Eq. 219, we see that $\epsilon(x)$ is at most quadratic in the coordinates. We therefore write the general expression

$$\epsilon(x)_\mu = a_\mu + b_{\mu\nu} x_\nu + c_{\mu\nu\rho} x_\nu x_\rho \tag{223}$$

Thus, we have four kinds of transformations:

- translation: $\epsilon_\mu = a_\mu$, i.e. ordinary translations independent of x.
- rotaion: $\epsilon_\mu = \omega_{\mu\nu} x_\nu$
- dilatation: $\epsilon_\mu = \lambda x_\mu$
- special conformal transformation: $\epsilon_\mu = b_\mu x^2 - 2 x_\mu b_\nu x_\nu$

Question: What is the meaning of special conformal transformation?

Conformal generators

Locally, the above conformal transformation can be expressed as the conformal generators: $a_\mu \partial_\mu, \omega^\mu_\nu x^\nu \partial_\mu x_\mu, \lambda x \cdot \partial, b^\mu (x^2 \partial_\mu - 2 x^\mu x \cdot \partial)$, where the number of them are respectively $d, d(d-1)/2, 1, d$. Thus, the total number generators for conformal symmetry is $(d+1)(d+2)/2$.

The generators satisfy the commutation relations:

$$P_\mu = -i\partial_\mu, L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu), D = -i x_\mu \partial_\mu, K_\mu = -i(2 x_\mu x^\nu \partial_\nu - x^2 \partial_\mu) \tag{224}$$

$$[D, P_\mu] = i P_\mu, [D, K_\mu] = -i K_\mu, [P_\rho, L_{\mu\nu}] = i(\eta_{\mu\nu} P_\nu - \eta_{\rho\nu} P_\mu), ... \tag{225}$$

Up to now, we consider the conformal transformation on the coordinates only. Here we consider how a quantum field changes under a given conformal transformation T_a: $\phi(x) \to \phi'(x') = (1 - i\epsilon_a T_a)\phi(x)$.

Some care should be taken when there is some special requirements on the quantum fields. For example, consider Lotentz group $L_{\mu\nu}\phi(0) = S_{\mu\nu}\phi(0)$. From Hausdorff formula $(e^{-A} B e^A = B + [B, A] + \frac{1}{2!}[[B, A], A] + ...)$:

$$e^{i x^\lambda P_\lambda} L_{\mu\nu} e^{-i x^\lambda P_\lambda} = L_{\mu\nu} - x_\mu P_\nu + x_\nu P_\mu \tag{226}$$

Thus we have

$$P_\mu \phi(x) = -i\partial_\mu \phi(x) \tag{227}$$

$$L_{\mu\nu}\phi(x) = i(x_\mu \partial_\nu - x_\nu \partial_\mu)\phi(x) + S_{\mu\nu}\phi(x) \tag{228}$$

Examples of conformal invariance

1) The simplest example of classical conformal symmetry is Maxwell's equations in the absence of sources

$$\partial_\mu F_{\mu\nu} = 0 \tag{229}$$

or,

$$\nabla^2 \vec{E} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla^2 \vec{B} = \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2}$$

2) Massless scale field. Like the electromagnetic field equations above, the equation of motion for this theory is also a wave equation,

$$\nabla^2 \varphi - \frac{\partial^2 \varphi}{\partial t^2} = 0 \tag{230}$$

and is invariant under the transformation $x \to \lambda x$,$t \to \lambda t$. The name massless refers to the absence of a term $\propto m^2\varphi$ in the field equation. Such a term is often referred to as a 'mass' term, and would break the invariance under the above transformation.

3) The above field equations in the examples are all linear in the fields. One non-linear equation example is $\phi - 4$ theory:

$$\nabla^2 \phi - \frac{\partial^2 \phi}{\partial t^2} + g\phi^3 = 0 \tag{231}$$

and is invariant under the transformation $x \to \lambda x$,$t \to \lambda t, \phi \to \lambda^{-1}\phi$. The key point is that the parameter g must be dimensionless, otherwise one introduces a fixed length scale into the theory: For this theory, this is only the case in $D = 4$. Note that under these transformations the argument of the function ϕ is unchanged.

Primary Fields

Under a local conformal map $z \to w(z)$, $\bar{z} \to w(\bar{z})$, a primary field is defined by the transformation

$$\phi'(w,\bar{w}) = \left(\frac{\partial w}{\partial z}\right)^{-h} \left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^{-\bar{h}} \phi(z,\bar{z}) \tag{232}$$

The above shows that a primary field of conformal dimensions $(h,\bar{h})$ transforms like the component of a covariant tensor of rank $h+\bar{h}$ having h "z" indices and $\bar{h}$ "$\bar{z}$" indices.

Please note that, this is equivalent to require the "line element" unchanged:

$$\phi(z,\bar{z})dz^h d\bar{z}^h = \phi(w,\bar{w})dw^h d\bar{w}^h = \phi(w,\bar{w})dz^h d\bar{z}^h (\frac{\partial w}{\partial z})^h (\frac{\partial \bar{w}}{\partial \bar{z}})^{\bar{h}} \tag{233}$$

If Eq. 232 holds *only* for global conformal transformations $w(z) \in SL(2,C)/Z_2$, we call this field as a quasi-primary field. Note that a primary field is always quasi-primary but the reverse is not true. Furthermore, not all fields in a CFT are primary or quasi-primary. Those fields are called secondary fields. (As we shall see, an example of a quasi- primary field that is not primary is the energy-momentum tensor.)

If the map $z \to w = z + \epsilon(z)$ is close to the identity, the variation of the field is

$$\begin{aligned} &(\frac{\partial w}{\partial z})^h = 1 + h\partial_z \epsilon(z) + O(\epsilon^2) \\ &\phi(z+\epsilon(z),\bar{z}) = \phi(z,\bar{z}) + \epsilon(z)\partial_z \phi(z,\bar{z}) \end{aligned} \tag{234}$$

and

$$\begin{aligned} &\delta_{\epsilon,\bar{\epsilon}}\phi(z,\bar{z}) = \phi'(z,\bar{z}) - \phi(z,\bar{z}) \\ =\ &\phi'(w-\epsilon(z),\bar{w}-\epsilon(\bar{z})) - \phi(z,\bar{z}) = \phi'(w,\bar{w}) - \epsilon(z)\partial_z\phi'(w) - \epsilon(\bar{z})\partial_{\bar{z}}\phi'(\bar{w}) - \phi(z,\bar{z}) \\ =\ &\left(\frac{\partial w}{\partial z}\right)^{-h} \left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^{-\bar{h}} \phi(z,\bar{z}) - \epsilon(z)\partial_z\phi'(w) - \epsilon(\bar{z})\partial_{\bar{z}}\phi'(\bar{w}) - \phi(z,\bar{z}) \\ \approx\ &-[(h\partial_z\epsilon + \epsilon\partial_z) + (\bar{h}\partial_{\bar{z}}\epsilon(\bar{z})) + \epsilon(\bar{z})\partial_{\bar{z}}]\phi(z,\bar{z}) \end{aligned} \tag{235}$$

Constraints on correlation functions

The objects of interest in quantum field theories are n-point correlation functions which are usually computed in a perturbative approach via either canonical quantisation or the path integral method. In this section, we will see that the exact two- and three-point functions for certain fields in a conformal field theory are already determined by the symmetries. This will allow us to derive a general formula for the OPE among quasi-primary fields.

Now the 2-point function $G(z_1, z_2) = \langle\phi_1(z_1)\phi_2(z_2)\rangle$ is supposed to satisfy infinitesimal conformal transformation, giving

$$0 = \delta_\epsilon G(z_1, z_2) = \langle\delta_\epsilon\phi_1(z_1)\phi_2(z_2)\rangle + \langle\phi_1(z_1)\delta_\epsilon\phi_2(z_2)\rangle \tag{236}$$

$$= \{[(h\partial_z\epsilon + \epsilon\partial_z) + (\bar{h}\partial_{\bar{z}}\epsilon(\bar{z})) + \epsilon(\bar{z})\partial_{\bar{z}}]|_{z=z_1} \tag{237}$$

$$+ [(h\partial_z\epsilon + \epsilon\partial_z) + (\bar{h}\partial_{\bar{z}}\epsilon(\bar{z})) + \epsilon(\bar{z})\partial_{\bar{z}}]|_{z=z_2}\}G(z_1, z_2) \tag{238}$$

We can write our infinitesmal conformal transformations as $\epsilon(z) = c_{-1} + c_0 z + c_1 z^2$. 1) If we use $\epsilon(z) = 1$ (translation symmetry), the condition of $(\partial_{z_1} + \partial_{z_2})\langle\phi_1(z_1)\phi_2(z_2)\rangle = 0$ gives green function depends only on $z_{12} = z_1 - z_2$: $\langle\phi_1(z_1)\phi_2(z_2)\rangle \sim f(z_1 - z_2)$. 2) $\epsilon(z) = z$ requires, $(h_1 + h_2 + z_1\partial_{z_1} + z_2\partial_{z_2})\langle\phi_1(z_1)\phi_2(z_2)\rangle = 0, \to (h_1 + h_2)f(z_{12}) + (z_1 - z_2)f'(z_{12}) = 0$, so the green function has the form of $f(z_{12}) \sim C_{12} z_{12}^{-h_1-h_2}\bar{z}_{12}^{-\bar{h}_1-\bar{h}_2}$. 3) $\epsilon(z) = z^2$ requires $(2h_1 z_1 + 2h_2 z_2 + z_1^2\partial_{z_1} + z_2^2\partial_{z_2})\langle\phi_1(z_1)\phi_2(z_2)\rangle = 0, \to \frac{C_{12}(h_1-h_2)}{(z_1-z_2)^{h_1+h_2+1}} = 0$ to require $h_1 = h_2 = h$, $\bar{h}_1 = \bar{h}_2 = h$. Therefore, that the 2-point function is constrained to take the form

$$G(z_1, z_2) = \frac{C_{12}}{z_{12}^{2h}\bar{z}_{12}^{2\bar{h}}} = \frac{C_{12}}{|z_{12}|^{\Delta}} \tag{239}$$

where $\Delta = 2h$.

Under the same method, we can also obtain the general form of three-point green's function and four-point green's function.

Conformal transformation in two dimension

CFT in 2D has very different properties. 2D CFT is Virasoro algebra, which is the central extension of Witt algebra.

1. Holomorphic function

In two dimension, Eq. 219 reduces to

$$\partial_1\epsilon_1 = \partial_2\epsilon_2, \quad \partial_1\epsilon_2 = -\partial_2\epsilon_1 \tag{240}$$

This is famous Cauchy-Riemann condition, which is sufficient conditions for holomorphic functions. (A complex function satisfying the Cauchy-Riemann equations is a holomorphic function in some open set.) Two dimensional conformal transformations thus coincide with the analytic coordinate transformations $z- > f(z) = z + \epsilon(z)$ and $\bar{z}- > f(\bar{z}) = \bar{z} + \epsilon(\bar{z})$.

That is, in 2D (on complex plane), all analytic or holomorphic transformations coincide with the conformal transformations. note that, there are infinite number of such kinds of holomorphic transformations in 2D, which is important for 2D (compared to $d > 2$ dimension).

2. Witt algebra

To calculate the commutation relations of the generators of the conformal algebra, i.e. infinitesimal transformations of the form $f(z)$, we take for basis

$$z \to z' = z + \epsilon(z), \quad \bar{z} \to \bar{z}' = \bar{z} + \epsilon(\bar{z}) \tag{241}$$

where take a Laurent expansion

$$\epsilon(z) = \sum_{n=-\infty}^{\infty} c_n z^{n+1}, \epsilon(\bar{z}) = \sum_{n=-\infty}^{\infty} c_n \bar{z}^{n+1} \tag{242}$$

and the c_n are assumed to be infinitesmal.

To get the infiniteesimal generators for 2D CFT, let us compute the action on functions

$$e^{c_n \ell_n + \bar{c}_n \bar{\ell}_n} \psi(z, \bar{z}) = \psi(z - \epsilon(z), \bar{z} - \bar{\epsilon}(\bar{z})) \tag{243}$$

which is equivalent to

$$(1 + c_n \ell_n + \bar{c}_n \bar{\ell}_n)\psi(z, \bar{z}) = (1 - \epsilon(z)\partial_z - \bar{\epsilon}(\bar{z})\partial_{\bar{z}})\psi(z, \bar{z}) \tag{244}$$

So the generators of the conformal algebra, or the corresponding infinitesimal generators are $\ell_n = -z^{n+1}\partial_z$ and $\bar{\ell}_n = -\bar{z}^{n+1}\partial_{\bar{z}}$. They satisfy the algebras

$$[\ell_m, \ell_n] = z^{m+1}\partial_z(z^{n+1}\partial_z) - z^{n+1}\partial_z(z^{m+1}\partial_z) = (n+1)z^{m+n+1}\partial_z - (m+1)z^{m+n+1}\partial_z = (m-n)\ell_{m+n}, \tag{245}$$

$$[\bar{\ell}_m, \bar{\ell}_n] = (m-n)\bar{\ell}_{m+n}, \tag{246}$$

$$[\ell_n, \bar{\ell}_m] = 0 \tag{247}$$

The first and second equations are copies of the Witt algebra. In the quantum case, the above algebras will be corrected to include an extra term proportional to a central charge. Since two independent algebras naturally arise, it is frequently useful to regard z and $\bar{z}$ as independent coordinates.

Remark. 1) Treating z and $\bar{z}$ as independent variables yields two copies of the Witt Algebra $A \oplus \bar{A}$. 2) Imposing the 'physical condition' leaves us with the subalgebra of $A \oplus \bar{A}$ generated by $\ell_n + \bar{\ell}_n$ and $i(\ell_n - \bar{\ell}_n)$ for all n. 3) For a Quantum Theory, we need a central extension of the Witt Algebra, which is the so-called Virasoro Algebra.

3. Global conformal transformations

The above algebra is defined locally, which may be ill-defined on some singular points.

Holomorphic conformal transformations are generated by vector fields

$$v(z) = \sum_n a_n \ell_n = \sum_n a_n z^{n+1} \partial_z \tag{248}$$

Non-singularity of $v(z)$ as $z \to 0$ allows $n \geq -1$. Similarly, non-singularity at $z \to \infty$ requires $n \leq 1$. Thus, we see that only the conformal transformations generated by a ℓ_n for $n = 0, \pm 1$ are globally defined.

In two dimensions the global conformal group is defined to be the group of conformal transformations that are well-defined and invertible on the Riemann sphere $C \cup \infty$. How does this finite subalgebra correspond to the momentum, rotation, etc. generators?

It is clear that $\ell_{-1} = \partial_z = \frac{1}{2}(\partial_0 - i\partial_1)$ and $\bar{\ell}_{-1} = \frac{1}{2}(\partial_0 + i\partial_1)$ generate translations on the complex plane. Similarly, ℓ_1 and $\bar{\ell}_1$ generate special conformal translations (to see it, make the transformation of $w- > z^{-1}$, then l_1 is the translational symmetry on w). To understand ℓ_0, we can consider complex polar coordinates $z = r\exp^{i\theta}$. In terms of these variables, $\ell_0 = r\partial_r - i\partial_\theta$, $\bar{\ell}_0 = r\partial_r + i\partial_\theta$. Then the useful linear combinations are easily seen to be $\ell_0 + \bar{\ell}_0 = r\partial_r$, $i(\ell_0 - \ell_0) = \partial_\theta$. Therefore, for definition, we identify ℓ_{-1} and $\bar{\ell}_{-1}$ is the translational operator, $i(\ell_0 - \bar{\ell}_0)$ as rotation operator and $\ell_0 + \bar{\ell}_0$ as dilation operator.

- translation: $a = 1, b, c = 0, d = 1$, $l_{-1} + \bar{l}_{-1}, i(l_{-1} - \bar{l}_{-1})$
- rotation: $a = e^{i\theta/2}, b = 0, c = e^{-i\theta/2}, d = 0$, i$(l_0 - \bar{l}_0)$
- dialtation: $a = \lambda, b = 0, c = \lambda^{-1}, d = 0$, $l_0 + \bar{l}_0$
- special conformal transformation: $a = 1, b = 0, c, d = 1$, $l_1 + \bar{l}_1, i(l_1 - \bar{l}_1)$

Together, these operators generate transformations of the form $z \to \frac{az+b}{cz+d}, ad - bc = 1$ ($\in SL(2, C)/Z_2 = SO(3, 1)$ on Riemann sphere $C \cup \infty$), which is precisely the conformal

group (or Mobius group) $PSL(2,C)$ (The quotient by Z_2 is due to the fact that $z \to \frac{az+b}{cz+d}$ is unaffected by taking all of a, b, c, d to minus themselves; For this transformation to be invertible, it requires $|ad - bc| \neq 0$, then it to get $ad - bc = 1$).

The distinction encountered here between global and local conformal groups is unique to two dimensions (in higher dimensions there exists only a global conformal group).

4. Virasoro algebra from central extension

Here we discuss the central extension of Witt algebra. By introducing the central extension of Witt algebra (or said in group level, a central extension of the group PSL(2, C)). Why we need to consider such a group extension? Remaind that the transformation we considered is projective, in physics, this means that the state $|\psi\rangle$ is indistinguishable from any nonzero scalar multiple $c|\psi\rangle$. In general, the projective representation cannot be lifted to a true representation (ordinary linear representation), but the central extension allows you to write a true representation of a different group (in the case of the original group is a Lie group, in our case, the group is SL(2, C)) which is a Lie group). In our case, the Witt algebra becomes the Virasoro algebra (We leave the derivation of this relation for future)

$$[L_n, L_m] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}. \tag{249}$$

where $L_n (n \in Z)$ denotes the elements of the central extension of the Witt algebra, and c is central charge. This is the most important results for 2D CFT. Crucially, the study of 2D CFT is actually equivalent to analysis of representation of this infinite Virasoro algebra.

5. Radial quantization

To probe more carefully the consequences of conformal invariance in a two dimensional quantum field theory, we enter into some of the details of the quantization procedure (we will focus our studies on conformal field theories defined on Euclidean two-dimensional flat space.). We begin with flat Euclidean "space" and "time" coordinates σ_1 and σ_0 (In Minkowski space, one needs to make a Wick rotation $\sigma_0 = it$). The coordinate $\sigma_1 = \sigma_1 + 2\pi$, is the periodic boundary condition for 1d spatial dimension (Or, it is to eliminate any infrared divergences, we compactify the space coordinate). Thus, we define the complex number $\zeta = \sigma_0 + i\sigma_1$.

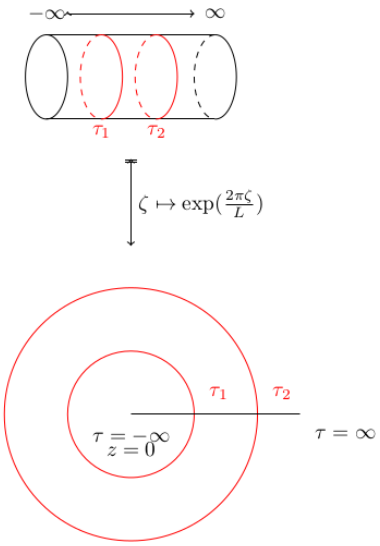


FIG. 8: Map from cylinder to plane.

The conformal map

$$z = exp(\sigma_0 + i\sigma_1) \tag{250}$$

maps the cylinder to the complex plane. Equal time surfaces, $\sigma_0 = const$, become circles of constant radius on the z-plane, and time reversal, $\sigma_0 \to -\sigma_0$, becomes $z \to 1/z^*$. Then the time translation $\sigma_0 -> \sigma_0 + T$ becomes dilatation on the complex plane $z -> e^T z$. So the dilatation generator on the conformal plane can be regarded as the Hamiltonian for the system, and the Hilbert space is built up on surfaces of constant radius. This procedure for defining a quantum theory on the plane is known as radial quantization (Hilbert state space defined on circles about the origin, and propagation of states in the radial direction dilation operator is the Hamiltonian; rotation operator is a spatial translation). Thus the hamiltonian on the complex plane is

$$H = (\ell_0 + \bar{\ell}_0). \tag{251}$$

The radial quantization, in particular, is to quantize this Hamiltonian or said to quantize the generators on the complex plane. And the quantized hamiltonian becomes

$$H = (L_0 + \bar{L}_0). \tag{252}$$

Application: Transverse Ising chain

Let us recall the Virasoro algebra, the operator $L_n(\bar{L}_n)$ satisfy (see Eq. **??**):

$$L_n = \frac{1}{2\pi i}\int z^{n+1}T(z)dz \tag{253}$$

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta(n_m, 0) \tag{254}$$

where $T(z)$ is energy-momentum tensor, c is central charge.

Now we consider a one-dimension quantum spin chain, with periodic boundary condition. This is related to the infinite-long stripe with a finite-size width N in two-dimension, which can be obtained by the conformal transformation from

$$w = x + iy = \frac{N}{2\pi}\ln z \tag{255}$$

which maps the whole z-plane into the strip $|Imw| < 2\pi$ with periodic condition.

The transformation of energy-momentum tensor is (Eq. **??**):

$$T'(w) = (\partial_z w)^{-2}[T(z) - \frac{c}{12}S(w,z)] = (\frac{2\pi}{N})^2[z^2T(z) - \frac{c}{24}] \tag{256}$$

Please note that, the traceless condition is violated after the transformation. This is the chiral anomaly. It goes to zero, only when $N \to \infty$.

Hence the conformal invariant Hamiltonian is

$$H = \frac{1}{2\pi}\int_0^N dy[T'(w) + \bar{T}'(\bar{w})] = \frac{2\pi}{N}(L_0 + \bar{L}_0) - \frac{\pi c}{6N} + K \tag{257}$$

where K is a constant. Please note that this Hamiltonian is like Eq. 252.

Then we use the state $|\Delta\rangle$, $L_0|\Delta+n\rangle = (\Delta+n)|\Delta+n\rangle, \bar{L}_0|\bar{\Delta}+n\rangle = (\bar{\Delta}+n)|\bar{\Delta}+n\rangle$. We got the eigenvalues as

$$E = \frac{2\pi}{N}(\Delta + \bar{\Delta} + n + \bar{n}) - \frac{\pi c}{6N} + K, \tag{258}$$

$$E_0 = -\frac{\pi c}{6N} + K \tag{259}$$

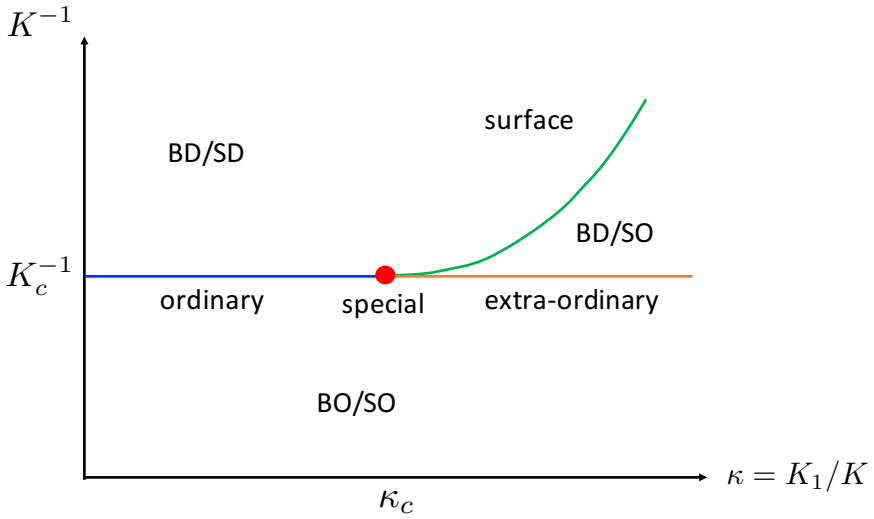


FIG. 9: Phase diagram of the classical $O(N)$ model with a boundary in dimension $d > 3$. BO stands for bulk ordered, SO - surface ordered, BD - bulk disordered, SD - surface disordered. For $d = 3$ and $N = 1$ the phase diagram is the same. For $d = 3$ and $N = 2$ the phase diagram has the same topology, but the BD/SO region only has quasi-long-range surface order.

SURFACE CRITICALITY

The classical $O(N)$ model in dimension $d \geq 3$ at its bulk critical point admits three boundary universality classes: the ordinary, the extra-ordinary and the special. For the ordinary transition the bulk and the boundary order simultaneously; the extra-ordinary fixed point corresponds to the bulk transition occurring in the presence of an ordered boundary, while the special fixed point corresponds to a boundary phase transition between the ordinary and the extra-ordinary classes.

Consider the classical lattice model in d-dimensions

$$\beta H = -\sum_{\langle ij \rangle} K_{ij} \vec{S}_i \cdot \vec{S}_j \,. \tag{260}$$

and let the boundary be at $x_d = 0$. We will be mostly interested in the case $d = 3$, but will keep $2 < d < 4$ general for now. We wish to study this model at its bulk critical point $K = K_c$ and in the $\kappa \gg 1$ region, when the surface has a strong tendency to local order. First, imagine turning off the couplings connecting the outermost surface layer to the next layer. The system without the outermost surface layer is then expected to realize the *ordinary* boundary universality class. We call the corresponding continuum fixed-point

bulk+boundary action of the d-dimensional $O(N)$ model, $S_{ordinary}$. We denote the bulk order parameter of the $O(N)$ model by $\vec{\phi}(x, x_d)$. The initially decoupled outermost surface layer can be described by the $d-1$ dimensional continuum $O(N)$ non-linear σ-model for the field $\vec{n}$,

$$S_n = \int d^{d-1}x \left(\frac{1}{2g} (\partial_\mu \vec{n})^2 - \vec{h} \cdot \vec{n} \right), \quad \vec{n}^2 = 1 \,. \tag{261}$$

Here, $\vec{h}$ is a small symmetry breaking field that will be used as an infra-red regulator. When $\kappa \gg 1$, we expect g to be small.

Now, let's restore the coupling of the outermost surface layer to the next layer: in the continuum description, we expect a coupling

$$S_{n\phi} = -\tilde{s} \int d^{d-1}x \, \vec{n}(x) \cdot \vec{\phi}(x, x_d = 0) \,. \tag{262}$$

to be generated. Here $\vec{\phi}(x, x_d = 0)$ should be understood as the lowest dimension $O(N)$ vector boundary operator of the $O(N)$ model at its ordinary boundary fixed point. Thus, we study the action

$$S_{UV} = S_{ordinary} + S_n + S_{n\phi}. \tag{263}$$

We want to understand what are the effects of the coupling $\tilde{s}$. To do so, we will work around the fixed point $g = 0$. When g is strictly zero, the fluctuations of $\vec{n}$ are frozen. Let's choose $\vec{n}$ to point along the N-th direction. The coupling $S_{n\phi}$ then acts as a boundary symmetry breaking field for the bulk $O(N)$ model. Such a field is relevant at the ordinary boundary fixed point and makes the boundary flow to the so-called "normal" fixed point.

While the action (263) provides a conceptually clear $O(N)$ symmetric regularization of the model we wish to consider, it is inconvenient to work with. Indeed, even at $g = 0$ we don't know the details of the flow from the ordinary to the normal boundary fixed point of the $O(N)$ model. Thus, we'd like to start with the end-point of this flow. We consider

$$S_{IR} = S_{normal} + S_n - s \int d^{d-1}x \, \pi_i(x) \hat{\phi}_i(x) + \delta S, \tag{264}$$

where $\vec{n} = (\vec{\pi}, \sigma = \sqrt{1 - \vec{\pi}^2})$ and S_{normal} is the conformal fixed point of the $O(N)$ model with a normal boundary (and the symmetry-breaking field pointing along the Nth direction). $\hat{\phi}_i$ is the boundary $O(N-1)$ vector at the normal fixed point.

FIG. 10: The Feymann diagram of surface correction contributing to the Green's function.

Renormalization Group analysis

We now perform RG on the model (264). Since the coupling s has been fixed by symmetry, only the coupling g is allowed to run. As in the standard $O(N)$ model near 2d we let

$$\frac{dK}{d\ell} = (d-2)K - (n-2)S^d\Lambda^{d-2} \leftrightarrow \frac{dT}{d\ell} = -\frac{dK}{K^2 d\ell} = -(d-2)T + (n-2)S^d\Lambda^{d-2}T^2 \tag{265}$$

which is from

$$\begin{aligned} -\beta H = &-\frac{K(1+\frac{I_d(b)}{K})b^{d-2}\zeta^2}{2}\int d^dx'(\nabla\pi'(\mathbf{x}'))^2 \\ &-\frac{Kb^{d-2}\zeta^4}{2}\int d^dx'(\pi'(\mathbf{x}')\nabla\pi'(\mathbf{x}'))^{\mathbf{2}} + \frac{\rho\zeta^{\mathbf{2}}}{\mathbf{2}}\int \mathbf{d^dx'}(\pi(\mathbf{x}'))^{\mathbf{2}} \end{aligned} \tag{266}$$

and

$$\begin{aligned} &K' = b^{d-2}\zeta^2 K(1+\frac{I_d(b)}{K}) = b^{d-2}[1-\frac{(n-1)}{2}\frac{I_d(b)}{K}]^2K(1+\frac{I_d(b)}{K}) \approx b^{d-2}K[1-\frac{n-2}{K}I_d(b)] \\ &\Rightarrow K+dK = (1+d\ell)^{d-2}K[1-\frac{n-2}{K}S_d\Lambda^{d-2}d\ell] \end{aligned} \tag{267}$$

Next we calculate the correction due to coupling to $\hat{\phi}_i(x)$.

We will denote the full π propagator by $D = \langle\pi_i(p)\pi_i(-p)\rangle$. Going to momentum space the first order Green's function (see Fig.):

$$\delta_s D(p) = s^2D_0^2(p)\int d^{d-1}z\langle\hat{\phi}_i(z)\hat{\phi}_i(z)\rangle_0 e^{-i\vec{p}\cdot\vec{z}} = s^2D_0^2(p)\int d^{d-1}z\,\frac{1}{|z|^{2(d-1)}}e^{-i\vec{p}\cdot\vec{z}}. \tag{268}$$

where we used the scaling dimension of $\hat{\phi}_i$ is $d-1$ (Noether current operator due to the continuous symmetry). Alternatively, letting the self-energy $\Sigma_\pi(p)$ be defined as $D(p)^{-1} = D_0(p)^{-1} + \Sigma_\pi(p)$,

$$\Sigma_\pi(p) \sim s^2\int d^{d-1}z\,\frac{1}{|z|^{2(d-1)}}e^{-i\vec{p}\cdot\vec{z}}. \tag{269}$$

We notice that $\delta_s\Sigma_\pi(p=0,h=0) \neq 0$. This would lead to the breaking of $O(N)$ rotational symmetry. Thus, to restore the $O(N)$ symmetry, we add a counterterm $\delta S =$

$C\int d^{d-1}x\,\vec{\pi}^2(x)$, with $C=-\delta_s\Sigma_\pi(p=0)$. Following this,

$$\Sigma_\pi(p)\to s^2\int d^{d-1}z\,\frac{1}{|z|^{2(d-1)}}\left(1-e^{-i\vec{p}\cdot\vec{z}}\right). \tag{270}$$

Setting $d=3$ and performing the integral we obtain to logarithmic accuracy,

$$\Sigma_\pi(p)=\frac{\pi s^2}{2}p^2\ln\frac{L}{a}=\frac{\pi s^2}{2}p^2\ln b=\frac{\pi s^2}{2}p^2\ln\frac{\Lambda}{\Lambda/b}. \tag{271}$$

Therefore the full Green's function becomes

$$D(p)=\frac{1}{D_0(p)^{-1}+\Sigma_\pi(p)}=\frac{1}{Kp^2+\frac{\pi s^2}{2}p^2\ln b} \tag{272}$$

$$\begin{aligned}
&\int d^2z\frac{e^{-i\vec{p}\cdot\vec{z}}}{|z|^4}\\
&=\int zdzd\theta\frac{e^{-ipz\sin\theta}}{z^4}=2\pi\int dz\frac{J_0(pz)}{z^3}=\int d(pz)p^{-1}\frac{J_0(pz)p^3}{(pz)^3}=2\pi p^2\int dt\frac{J_0(t)}{t^3}\\
&\int d^2z\frac{1-e^{-i\vec{p}\cdot\vec{z}}}{|z|^4}=\\
&=2\pi p^2\int dt\frac{1-J_0(t)}{t^3}\overset{t\ll1}{\to}2\pi p^2\int dt\frac{\frac{t^2}{4}}{t^3}=\frac{\pi p^2}{2}\int_{pa}^{Lp}dt\frac{1}{t}=\frac{\pi p^2}{2}\ln\frac{L}{a}
\end{aligned} \tag{273}$$

So we get modified coupling strength as

$$\begin{aligned}
K'&=b^{d-2}\zeta^2K(1+\frac{I_d(b)}{K})(1+\frac{\pi s^2}{2K}\ln b)\\
&=b^{d-2}[1-\frac{(n-1)}{2}\frac{I_d(b)}{K}]^2K(1+\frac{I_d(b)}{K})(1+\frac{\pi s^2}{2K}\ln b)\\
&\approx b^{d-2}K[1-\frac{n-2}{K}I_d(b)](1+\frac{\pi s^2}{2K}\ln(1+d\ell))\\
&\approx b^{d-2}K[1-\frac{n-2}{K}I_d(b)](1+\frac{\pi s^2}{2K}d\ell)\\
&\Rightarrow K+dK=(1+d\ell)^{d-2}K[1-\frac{n-2}{K}S_d\Lambda^{d-2}d\ell](1+\frac{\pi s^2}{2K}d\ell)\\
&\frac{dK}{d\ell}=(d-2)K-(n-2)S^d\Lambda^{d-2}+\frac{\pi s^2}{2}
\end{aligned} \tag{274}$$

Converting to the temperature

$$\begin{aligned}
\frac{dT}{d\ell}=-\frac{dK}{K^2d\ell}&=-(d-2)T+(n-2)S^d\Lambda^{d-2}T^2-\frac{\pi s^2}{2}T^2\\
&=-(d-2)T-(\frac{\pi s^2}{2}-(n-2)S^d\Lambda^{d-2})T^2
\end{aligned} \tag{275}$$

Phase diagram

What are the consequences of the RG analysis in Eq 275? Letting ℓ be the RG scale, we observe that the physics depends on the sign of $\alpha = (\frac{\pi s^2}{2} - (n-2)S^d\Lambda^{d-2})$. For $d = 2$, the RG equation reduces to

$$\frac{\partial T}{\partial \ell} = -\alpha T^2. \tag{276}$$

When $\alpha > 0$ the $T = 0$ fixed point is stable, while when $\alpha < 0$ it is unstable. Crucially, we known that when $N = 2$,

$$\alpha(N = 2) = \frac{\pi s^2}{2} > 0, \tag{277}$$

while for large (but finite N), from Eq. (302)

$$\alpha(N) \approx -(n-2) < 0, \quad N \to \infty. \tag{278}$$

Thus, there must be a critical N_c at which α switches sign. Naive extrapolation of the large-N result (278) gives $N_c \approx 4$ (by accounting the large-N result of $s^2(N \to \frac{N}{2\pi^2})$). Thus, for $2 \leq N < N_c$, the extra-ordinary fixed point survives in $d = 3$, but in a modified form with no true long range order and order parameter correlations that decay only as a power of logarithm. We refer to such behavior as the extra-ordinary-log universality class. In particular, this occurs for the case $N = 2$ where we are confident that $\alpha > 0$.

Also, from Eq. 276, we get the running of temperature as

$$T(\ell) = \frac{T_0}{1 + \alpha T_0 \ell} \tag{279}$$

Correlation

Let us try to calculate correlator in real space.

$$\begin{aligned} D(p) &= \frac{1}{D_0(p)^{-1} + \Sigma_\pi(p)} = \frac{1}{Kp^2 + \frac{\pi s^2}{2} p^2 \ln \frac{\Lambda}{p}} = \frac{1}{Kp^2(1 + \frac{\pi s^2}{2K} \ln \frac{\Lambda}{p})} \\ &= \frac{1}{Kp^2(1 + \frac{\alpha}{K}(1 + \frac{n-2}{\alpha}) \ln \frac{\Lambda}{p})} \approx \frac{1}{Kp^2(1 + \frac{\alpha}{K} \ln \frac{\Lambda}{p})^{(1+\frac{n-2}{\alpha})}} \\ \to D(x) &= \int d^2 p e^{-i\vec{p}\cdot\vec{x}} D(p) \stackrel{!}{=} \frac{1}{(\ln x)^{1+\frac{n-2}{\alpha}}} \end{aligned} \tag{280}$$

This integral is too difficult to deal with. Next we will a different way to get it.

The Green's function is defined as two-point correlator:

$$G(x;T) = \langle n(x)n(0)\rangle \tag{281}$$

which depends on the temperature (thinking about that it is close to but deviated away from the critical point). Next we take one step of RG transformation, we will get

$$G'(x';T') = \langle n'(x')n'(0)\rangle = \zeta^{-2}\langle n(x)n(0)\rangle = \zeta^{-2}G(x;T) \tag{282}$$

$$\Rightarrow G'(x';T') = ZG(x;T) \tag{283}$$

where $Z = \zeta^{-2}$ is renormalization factor. From $\zeta = 1 - \frac{n-1}{2}\frac{I_d(b)}{K} \approx 1 - \frac{n-1}{2K}S^d\Lambda^{d-2}\ell \equiv b^{-\Delta_n}$, we get the scaling dimension of operator n as

$$\Delta_n \approx \frac{n-1}{2K}S^d\Lambda^{d-2} \tag{284}$$

Then we assume that, at the fixed point, the two-point correlator should be a RG invariant:

$$\ell\frac{dG}{d\ell} = 0 \Rightarrow \quad \ell\frac{\partial G}{\partial \ell} + \ell\frac{\partial G}{\partial T}\frac{\partial T}{\partial \ell} + \ell\frac{\partial G}{\partial Z}\frac{\partial Z}{\partial \ell} = 0 \tag{285}$$

$$\ell\frac{\partial G}{\partial \ell} + \ell\frac{\partial G}{\partial T}\frac{\partial T}{\partial \ell} + \ell\frac{\partial \ln Z}{\partial \ell}G = 0 \tag{286}$$

The second term, $\frac{\partial T}{\partial \ell}$, is just the beta function, which should vanish at the fixed point. So the second term is zero. Then we get,

$$\frac{\partial G}{\partial \ell} + \frac{\partial \ln Z}{\partial \ell}G = 0 \rightarrow \frac{\partial G}{\partial \ell} + 2\Delta_n G = 0 \tag{287}$$

Next we recall the RG equation, $\partial\ell = -\frac{\partial T}{\alpha T^2}$ (see Eq. 276), and we will use it to get a new differential equation for G:

$$\frac{\partial G}{G} = -2\Delta_n \times (-\frac{\partial T}{\alpha T^2}) = \frac{n-1}{\alpha}\frac{\partial T}{T} = q\frac{\partial T}{T} \tag{288}$$

Here we set $q = \frac{n-1}{\alpha}$. By integral on both sides, we have

$$\ln G \sim q \ln T \Rightarrow G \sim T^q \tag{289}$$

Next plugging in the running of temperature $T(\ell) \sim \frac{T_0}{1+\alpha T_0 \ell}$:

$$G \sim (\frac{T_0}{1+\alpha T_0\ell})^q \sim \frac{1}{\ell^q} \sim \frac{1}{(\ln(x/x_0))^q} \tag{290}$$

Here we get the correlator at the extraordinary fixed point.

Fix the value of s

We now use the $O(N)$ symmetry to fix the value of s. If the symmetry breaking field points along the N-th direction, letting $\phi_N = \sigma$, we have the operator product expansion (OPE):

$$\sigma(x, x_d) \sim \frac{a_\sigma}{(2x_d)^{\Delta_\phi}} + \mu_\sigma (2x_d)^{d-\Delta_\phi} \hat{\sigma}(x) + \ldots, \quad x_d \to 0\,. \tag{291}$$

Here, a_σ and μ_σ are universal constants.

The boundary scaling dimension of the lowest $O(N-1)$ vector on the boundary $\hat{\phi}_i$, $i = 1 \ldots N-1$ is also known exactly: $\Delta_{\hat{p}hi_i} = d-1$. We write:

$$\phi_i(x, x_d) \sim \mu_\phi \, (2x_d)^{d-1-\Delta_\phi} \hat{p}hi_i(x) + \ldots, \quad x_d \to 0, \tag{292}$$

where μ_ϕ is a universal constant.

We continue to work at $g = 0$, where $\vec{n}$ is a classical frozen constant field. When this field points along the N-th direction, $\vec{n} = (\vec{0}, 1)$ we know that

$$\langle \sigma(x_d) \rangle = \frac{a_\sigma}{(2x_d)^{\Delta_\phi}}, \quad \langle \phi_i \rangle = 0\,. \tag{293}$$

If we rotate $\vec{n}$ by an infinitesimal angle α towards the direction $\hat{1}$, $n_1 = \sin\alpha, n_d = \cos\alpha$, we should get

$$\langle \phi_1(x_d) \rangle = \frac{a_\sigma \sin\alpha}{(2x_d)^{\Delta_\phi}}\,. \tag{294}$$

But, from (264), to first order in α,

$$\langle \phi_1(x_d) \rangle = s\alpha \int d^{d-1}x \, \langle \phi_1(x, x_d) \hat{\phi}_1(x) \rangle_{norm}, \tag{295}$$

where the subscript *norm* denotes the expectation value taken with respect to the action S_{normal}. The correlation function on the RHS is fixed by conformal symmetry.[Cardy, 1986] Indeed, we have

$$\langle \phi_i(x, x_d) \phi_j(x', x'_d) \rangle_{norm} = \frac{\delta_{ij}}{(4x_d x'_d)^{\Delta_\phi}} g(\xi), \quad \xi = \frac{|x-x'|^2 + (x_d - x'_d)^2}{4x_d x'_d}\,, \tag{296}$$

with $g(\xi)$ - a universal function. Our choice of normalization of ϕ^i in the absence of the boundary implies $g(\xi) \to \xi^{-\Delta_\phi}$, $\xi \to 0$. Furthermore, on both operators in the correlator (296) requires

$$g(\xi) \to \frac{\mu_\phi^2}{\xi^{d-1}}, \quad \xi \to \infty\,. \tag{297}$$

Now, on just one of the operators in (296) we obtain,

$$\langle \phi_i(x, x_d)\hat{\phi}_j(x')\rangle_{norm} = \mu_\phi \delta_{ij} \frac{(2x_d)^{d-1-\Delta_\phi}}{(|x-x'|^2+x_d^2)^{d-1}}\,. \tag{298}$$

Substituting this into (295) and taking the integral over x,

$$s = \frac{\Gamma(d-1)}{(4\pi)^{\frac{d-1}{2}}\Gamma(\frac{d-1}{2})}\frac{a_\sigma}{\mu_\phi} \overset{d=3}{=} \frac{1}{4\pi}\frac{a_\sigma}{\mu_\phi}\,. \tag{299}$$

Thus, the coupling s is fixed by the $O(N)$ symmetry in terms of the universal constants a_σ and μ_ϕ. Crucially, s is dimensionless. We also note that s is not small. Thus, we will not be performing perturbation theory in s, but rather in g.

We note that so far we've only restored the $O(N)$ symmetry to leading order in fluctuations of $\vec{n}$ (in particular, the above analysis was carried out to linear order in α only). We may need to add extra terms to the Lagrangian to restore the symmetry at higher orders. One example of such a term is $\delta L \sim \vec{\pi}^2\pi_i\hat{p}hi_i$. However, these higher order terms will not affect our analysis below.

The value of s plays an important role in what follows, so we pause to discuss various results for a_σ and μ_ϕ. Explicit expressions can be obtained for a_σ and μ_ϕ in the limit $N\to\infty$. We have computed the first corrections to these quantities in $1/N$ for $d=3$.

$$a_\sigma^2 = 2(N+1)\left(1-\frac{\eta}{2}\right)+O\left(\frac{1}{N}\right) \approx 2(N+0.865)+O\left(\frac{1}{N}\right), \tag{300}$$

$$\mu_\phi^2 = \frac{1}{4}\left(1+\frac{1}{N}\right)\left(1-\frac{\eta}{2}\right)+O\left(\frac{1}{N^2}\right) \approx \frac{1}{4}\left(1+\frac{0.865}{N}\right)+O\left(\frac{1}{N^2}\right), \tag{301}$$

$$s^2 = \frac{N}{2\pi^2}+O\left(\frac{1}{N}\right). \tag{302}$$

Here $\eta \approx \frac{8}{3\pi^2 N}$ is the bulk anomalous dimesnion of ϕ: $\Delta_\phi = (d-2+\eta)/2$.

One can also obtain expressions for a_σ^2 and μ_ϕ^2 in the $4-\epsilon$ expansion:

$$a_\sigma^2 = \frac{4(N+8)}{\epsilon}\left(1-\frac{N^2+31N+154}{(N+8)^2}\,\epsilon\right), \tag{303}$$

$$\mu_\phi^2 = \frac{1}{3}\left(1-\frac{N+9}{6(N+8)}\epsilon\right). \tag{304}$$

Unfortunately, the utility of Eqs. (303),(304), in $d=3$ is not clear. In fact, substituting $\epsilon=1$ into (303) gives a negative a_σ^2 for all N. On the other hand, substituting $\epsilon=1$ into (304) gives μ_ϕ^2 within 15% of the large-N estimate (302) for $N\geq 3$.

* Electronic address: zhuwei@westlake.edu.cn

[1] Assa Auerbach, Interacting electrons and quantum magnetism, Springer.

[2] E. Fradkin, Field Theories of Condensed Matter Physics. Cambridge University Press, 2013.

[3] Renormalization of the nonlinear σ model in $2+\epsilon$ dimensions, E. Brezin and J. Zinn-Justin, J. C. Le Guillou, Phys. Rev. B 14, 3110 (1976).

[4] Boundary criticality of the O(N) model in d = 3 critically revisited, Max A. Metlitski, SciPost Physics 12, 131(2022).

[5] E. Lieb, T. Schultz and D. Mattis, Annals of Physics, 16, 407 (1961)

[6] F. D. M. Haldane, Phys. Lett. A 93, 464 (1983); Phys. Rev. Lett. 50, 1153 (1983).