Lecture notes for
The Quantum Hall Effect and Beyond:
Fractional Quantum Hall Effect

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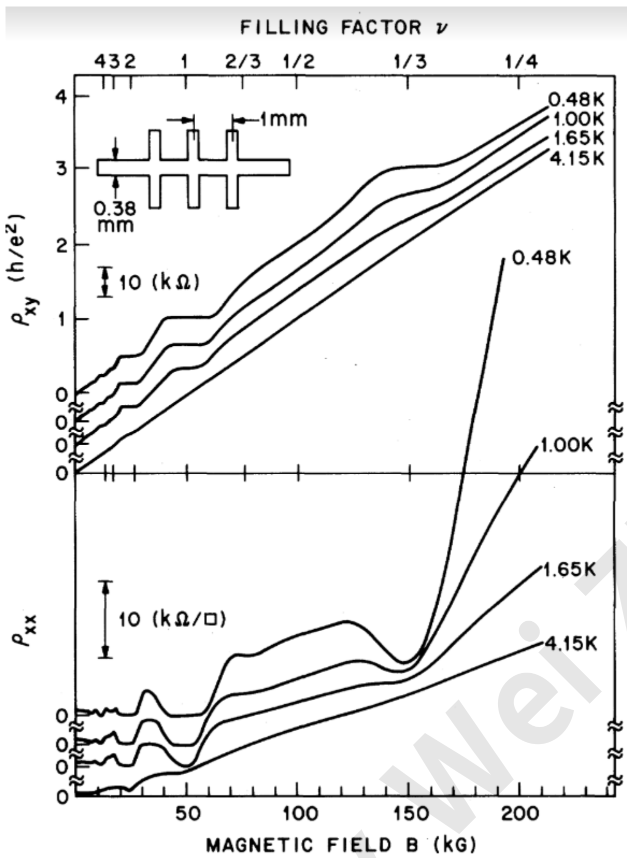

FIG. 1: The magnetic field dependence of Hall and longitudinal resistances, ρ_{xy} and ρ_{xx}, for a two-dimensional electron system at the GaAsAlGaAs interface. Source: D. C. Tsui, H. L. Stormer, and A. C. Gossard, Phys. Rev. Lett. 48, 1559 (1982).

EXPERIMENTAL OBSERVATIONS

Having just determined that the quantum Hall effect is some sort of spectroscopy on the charge of the electron, it was particularly surprising in 1982 when Dan Tsui and Horst Stormer1 discovered quantum Hall plateaus at fractional values of the filling fraction $\nu = p/q$, and the the Hall resistance

$$R_H = \frac{h}{\nu e^2} \tag{1}$$

with p and q integers. This effect is appropriately called the Fractional quantum Hall effect. The first plateau observed was the $\nu = 1/3$ plateau, but soon thereafter many more plateaus were discovered. The Nobel prize for this discovery was awarded in 1998.

An improvement of experimental conditions (higher mobilities, higher magnetic fields, lower temperatures) has led to the observation of a large number of fractions since then.

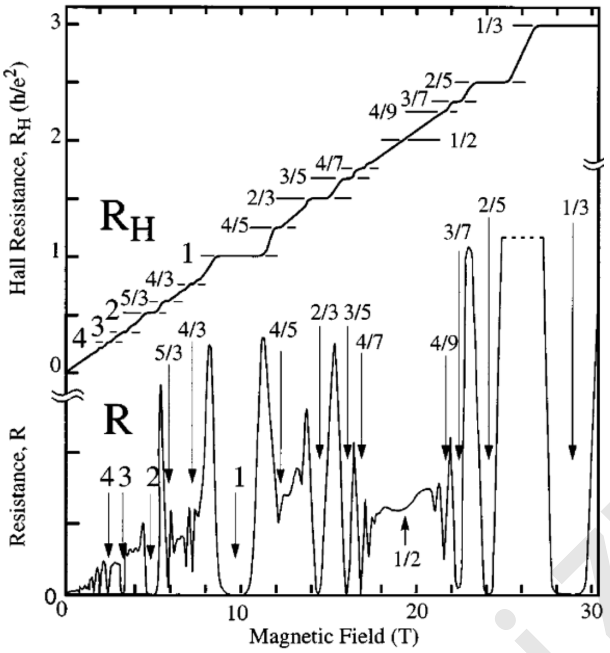

FIG. 2: The improved experimental data. Source: H. L. Stormer, Rev. Mod. Phys. 77, 875 (1999).

Nowadays (2020), the number of observed fractions, counting only fractions below unity, is more than 50, such as $\frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{5}{3}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \ldots$.

They are sensitive to disorder. Low-mobility samples do not show a FQHE. The FQHE has a characteristic energy scale of only a few degrees kelvin.

1. Why this phenomenon is exotic? The experimental observation indicates an incompressible state when a Landau level is partially filled. As with the integer case, disorder will be important in allowing us to have plateaus of finite width, but the fundamental physics of the fracitonal quantum Hall effect comes from the fact that we have a gapped incompressible systems at a particular filling fraction.

We restrict our attention to a clean system with a partially filled (say, 1/3 filled) Landau level. If there are Ne electrons in the system, there $3N_e$ available single electron orbitals in which to place these electrons. Thus in the absence of disorder, and in the absence of interaction, there are $\begin{pmatrix} 3N_e \\ N_e \end{pmatrix} \sim 6^{N_e}$, and all of these states have the same energy! In the thermodynamic limit this is an enormous degeneracy. (i.e. the energy gap should be zero at fractional filling.) This enormous degeneracy is broken by the interaction between the electrons, which will pick out a very small ground state manifold (in this case being just 3 degenerate ground states), and will leave the rest of this enormous Hilbert space with higher energy.

2. How to understand this phenomenon?

The disorder hypothesis may be immediately discarded as the driving mechanism of the FQHE because, in contrast to the IQHE, the FQHE only occurs in high- quality samples with low impurity concentrations.

No way to solve it in the single-particle picture. We have to envolve electron-electron interaction.

If the interaction is the only relevant scale, we thus obtain a system of strongly-correlated electrons for the description of which all perturbative approaches starting from the Fermi liquid are doomed to fail. The only hope one may have to describe the FQHE is then a well-educated guess of the ground state. This is quite different.

LAUGHLIN'S THEORY

Landau level in symmetric gauge

$$H = \frac{1}{2m}(\mathbf{p} - e\mathbf{A})^2 \tag{2}$$

where A(r) is the vector potential that generates the magnetic field

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{3}$$

When we choose the symmetric gauge: $\mathbf{A} = B(-y/2, x/2, 0)$,

$$H = \frac{1}{2}[(-i\partial_x - \frac{y}{2})^2 + (-i\partial_y + \frac{x}{2})^2] = \frac{1}{2}[-4\frac{\partial^2}{\partial z \partial \overline{z}} + \frac{1}{4}z\overline{z} - z\frac{\partial}{\partial z} + \overline{z}\frac{\partial}{\partial \overline{z}}] \tag{4}$$

where we define $z = x - iy = re^{-i\theta}, \overline{z} = x + iy = re^{i\theta}$, $\partial_x = \partial_z + \partial_{\bar{z}}, \partial_y = -i(\partial_z - \partial_{\bar{z}})$. The ladder operators can be defined as

$$a^\dagger = \tfrac{1}{\sqrt{2}}(\tfrac{\bar{z}}{2} - 2\tfrac{\partial}{\partial z}) \tag{5}$$

$$a = \tfrac{1}{\sqrt{2}}(\tfrac{z}{2} + 2\tfrac{\partial}{\partial \bar{z}}) \tag{6}$$

$$b^\dagger = \tfrac{1}{\sqrt{2}}(\tfrac{z}{2} - 2\tfrac{\partial}{\partial \bar{z}}) \tag{7}$$

$$b = \tfrac{1}{\sqrt{2}}(\tfrac{\bar{z}}{2} + 2\tfrac{\partial}{\partial z}) \tag{8}$$

$$[a, a^\dagger] = 1, \quad [b, b^\dagger] = 1 \tag{9}$$

The hamiltonian becomes

$$H = a^+a + 1/2 \tag{10}$$

In addition, The z component of the angular momentum operator is defined as

$$L_z = -i\hbar\frac{\partial}{\partial\theta} = -\hbar(z\frac{\partial}{\partial z} - \overline{z}\frac{\partial}{\partial \overline{z}}) = -\hbar(b^\dagger b - a^\dagger a) \tag{11}$$

Exploiting the property $[H, L_z] = 0$, the eigenfunctions are chosen to diagonalize H and L simultaneously. The eigenvalue of L is denoted by $m\hbar$; with this definition the quantum number m takes values $-n, -n+1, ...$

The ground state wave function is solved by $a|0,0\rangle = 0, b|0,0\rangle = 0$. We obtain

$$\langle r|0,0\rangle = \frac{1}{\sqrt{2\pi}\ell}e^{-\frac{|z|^2}{4}}. \tag{12}$$

The other wavefunctions are obtained by (The wave function in the lowest Landau level)

$$\langle r|n,m\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}\frac{(b^\dagger)^m}{\sqrt{m!}}\langle r|0,0\rangle \tag{13}$$

Especially, the single particle states are especially simple in the lowest Landau level (n = 0):

$$\langle r|n=0,m\rangle = \frac{(b^\dagger)^m}{\sqrt{m!}}\langle r|0,0\rangle = \frac{1}{\sqrt{2\pi 2^m m!}\ell} z^m e^{-|z|^2/4} \tag{14}$$

This wave function represents an electron localized circularly in disk. The maximum of the existence probability is on the circumference of a circle of radius $\sqrt{2m}\ell$, and the spread of the wave function in the radial direction is of the order of ℓ. The expectation value of r^2 is $\langle 0,m|r^2|0,m\rangle = 2(m+1)\ell^2$. Thus the largest value of m for which the state falls inside the disk is given by $m_{max} = \pi R^2/2\pi\ell^2$,

Exercises. Prove the normalization condition of electron wave functions.

$$\langle 0,m|0,m'\rangle = \int \overline{\frac{1}{\sqrt{2\pi 2^m m!}\ell} z^m e^{-|z|^2/4}} \frac{1}{\sqrt{2\pi 2^{m'} m'!}\ell} z^{m'} e^{-|z|^2/4} \tag{15}$$

$$= \frac{1}{\sqrt{2\pi 2^m m!}\ell}\frac{1}{\sqrt{2\pi 2^{m'} m'!}\ell}\int_{\psi=0}^{2\pi}\int_0^\infty dr d\psi e^{-i(m-m')\psi} r^{m+m'+1} e^{-r^2} \tag{16}$$

$$= \frac{\delta_{m,m'}}{m!} m! = \delta_{m,m'} \tag{17}$$

Exercises. Prove the average area of each Landau orbital.

Two-electron problem

Unsymmetrised two-particle basis states from the lowest Landau level have the form

$$\psi(z_1,z_2) \sim z_1^{l_1} z_2^{l_2} e^{-(|z_1|^2+|z_2|^2)/4} \tag{18}$$

with l non-negative integers. We will consider combinations of these that are eigenfunctions of relative and centre-of-mass angular momentum. They have the form

$$\psi(z_1,z_2) \sim (z_1-z_2)^l (z_1+z_2)^m e^{-(|z_1|^2+|z_2|^2)/4} \tag{19}$$

Laughlin wave function

When the FQHE was discovered, R. Laughlin realized that one could write down a many-body variational wave function at filling factor $\nu = 1/q$. This seminal idea opens a door the answer to the FQHE.

The many-body problem is

$$H = \sum_j [\frac{1}{2m}| - i\hbar\partial_j - e\mathbf{A}_j|^2] + \sum_{j<i} \frac{e^2}{|z_j - z_i|} \tag{20}$$

Laughlin proposed the wave function of ground state of the above hamiltonian at $\nu = 1/q$ is [R. Laughlin, Anomalous Quantum Hall Effect: An Incompressible Quantum Fluid with Fractionally Charged Excitations Phys. Rev. Lett. 50, 1395 (1983) ; ABSTRACT: This Letter presents variational ground-state and excited-state wave functions which describe the condensation of a two-dimensional electron gas into a new state of matter.]:

$$\psi_q = \prod_{j,i=1 i<j}^{N_e} (z_j - z_i)^q \prod_{j=1}^{N_e} e^{-|z_j|^2/4\ell^2} \tag{21}$$

Next we try to understand this famous wave function:

- For q being an odd integer, this wave function obeys Fermi statistics.
- The value q = 1 describes a full Landau level. Higher values ensure that the probability density falls faster to zero as a pair of particles approach each other.
- The angular momentum is $M = N_e(N_e - 1)q/2$.
- In the polynomial part of Laughlin's wave function, an electron coordinate z_i has $M = (N_e - 1)q$ as the maximum power. This is the maximum angular momentum that the electron can have, and in this state the area that this electron encloses becomes maximum. The maximum area is determined by the largest momentum: $A = 2(M + 1)\pi\ell^2$, thus the filling factor $\nu = N_e 2\pi\ell^2/A = N_e/(M+1) = N_e/[(N_e - 1)q] \sim 1/q$.

We emphasize once again that Laughlins wave function is not based on a mathematical derivation, although we will see below that there exist some mathematical models for which it describes the exact ground state, but it is more appropriately characterised as a variational wave function.

N_e	N_H	$\langle\Psi_0\|\Psi_3\rangle$
4	18	0.99804
5	73	0.99906
6	338	0.99644
7	1656	0.99636
8	8512	0.99540
9	45207	0.99406

FIG. 3: Overlap of the wave functions for a system at 1/3. Ψ_0 is the exact ground state of the Coulomb interaction.

Laughlin wave function builds in good correlations among the electrons because each electron sees an $q-$fold zero at he positions of all the other electrons. The wave function vanishes extremely rapidly if any two electrons approach each other, which helps to minimize the expectation value of the interaction.

How good the Laughlin wave function is? Let us see some numerics in Fig. 3, which compares the Laughlin wave function with numerically obtained exact diagonalization results.

Variational View Point

Alternatively, one can take q in Eq. 21 as a variational parameter. Then we can test which value of q will give the best variational energy. Consider Laughlins wave function as a function of the position z_k of some arbitrary but fixed electron k. There are $N-1$ factors of the type $(z_k - z_l)^q$, one for each of the remaining $N-1$ electrons. Now, remember that the highest power of the complex particle position is fixed by the number of states N_ϕ in each LL. This yields the relation $N_\phi = q(N-1)$. One notices that, in the thermodynamic limit, the "variational parameter" is entirely fixed by the filling factor $\nu = N/N_\phi = N/q(N-1) \approx 1/q$, thus $q = 1/\nu$. Since we need additional exchange relation, the even q should be excluded. Therefore, we only see FQHE at odd integer q.

Classical Plasma

We go further examine the physics behind this wave function. We think of the probability density arising from this wavefunction as if it were the Boltzmann weight for a problem in classical statistical mechanics. We define a fictitious inverse temperature $\beta = 1$ and classical Hamiltonian H_m via

$$|\psi_q|^2 = \exp[-H_m] \tag{22}$$

$$H_m = -2q \sum_{i<j} \ln |z_i - z_j| + \sum_j |z_j|^2/2\ell^2. \tag{23}$$

For a charge neutral two-dimensional classical plasma, the interaction is given by

$$V(r) = -e^2 \sum_{j<k} \ln r_{ij} + \frac{1}{2}\pi\rho e^2 \sum_j r_j^2 \tag{24}$$

where the particles are interacting via a two-dimensional Coulomb (logarithmic) interaction with each other and with a uniform neutralizing background. it is clear that H_m is the Hamiltonian for a two-dimensional classical plasma with, $e^2 = 2q, \rho_q = 1/(2\pi\ell^2 q)$. Therefore, in order to achieve charge neutrality, the plasma particles spread out uniformly in a disk with particle density corresponding to a filling factor $\nu = 1/q$, where q is an odd integer. The classical plasma provides strong support that the Laughlin state is indeed a translationally invariant liquid.

To interpret this form we should recall electrostatics in two dimensions: a point charge Q at the origin gives rise at radius r to an electric field

$$E(r) \sim \frac{Q}{2\pi r} with a potential V(r) \sim -\frac{Q}{2\pi} \ln r \tag{25}$$

Thus the two particle potential is like $-\frac{q^2}{2\pi} \ln |z_i - z_j|$ which represents the electrostatic interaction of particles with charge q.

The single particle term $\frac{q}{8\pi} \sum_k |z_k|^2$ would arise for particles of charge q moving in an electrostatic potential $|z|^2/(8\pi)$. We can view this potential as arising from a background charge distribution, and find the density of this charges using Poissons equation.

Quasi-hole Statistics and Fractionalization

We consider a situation where the filling factor is close to 1/q and there is only one quasihole in the system. Let us add a perturbation to a Hamiltonian which gives the fractional quantum Hall effect. The Hamiltonian then has a similar form to

$$H_{hole} = H_0 + \epsilon V(z - z_a) \tag{26}$$

The weak perturbation attracts the quasihole to z_a, which is a coordinate in the two-dimensional space represented by a complex number. The ground state of this Hamiltonian is evidently a state in which the quasihole is trapped at z_a:

$$\Psi(z_1, z_2, ..., z_{N_e}) \sim \prod_i (z_i - z_a)\psi_q \tag{27}$$

where ψ_q is the Laughlin wave function and there is an unspecified normalization factor here.

Now we move the quasihole in the real space and enclose a circle. The Berry phase in this case is calculated as follows:

$$\begin{aligned}
\gamma_C &= i \oint_C \langle \Psi_{r(t)} | \frac{\partial}{\partial r_t} \Psi_{r(t)} \rangle dr(t) \\
&= i \oint_C \langle \Psi_{z_a} | \frac{\partial}{\partial z_a} \Psi_{z_a} \rangle dz_a \\
&= i \oint_C dz_a \langle \Psi_{z_a} | [\frac{\partial}{\partial z_a} \sum_i \ln(z_i - z_a)] | \Psi_{z_a} \rangle \\
&= i \oint_C dz_a d^2 z \frac{\partial}{\partial z_a} \ln(z - z_a) \langle \Psi_{z_a} | \sum_i \delta(z - z_i) | \Psi_{z_a} \rangle \\
&= i \oint_C dz_a \int d^2 z \frac{\partial}{\partial z_a} \ln(z - z_a) \rho(z) \\
&= i \int d^2 z \rho(z) \oint_C dz_a \frac{\partial}{\partial z_a} \ln(z - z_a) \\
&= i \int d^2 z \rho(z)(-2i\pi) = 2\pi \int d^2 z \rho(z) = 2\pi \langle n \rangle = 2\pi S \frac{\nu}{2\pi \ell^2} = \frac{e}{q} \frac{BS}{\hbar}
\end{aligned} \tag{28}$$

Here we assume the electron density is uniform $\rho(z) = \nu / 2\pi\ell^2$. This result can be interpreted as the AB phase that a quasihole of charge e/q acquires in the magnetic field. The size of this charge coincides with that of the quasi-hole at $\nu = 1/q$.

Similarly, we can also study statistics of the quasiparticles. When the two quasiholes are at z_a, z_b, the wave function can be written as

$$\Psi_{a,b} = \prod (z_i - z_a)(z_i - z_b)\psi_q(z_1, z_2, ..., z_{N_e}) \tag{29}$$

And we calculate the berry phase, when the quasihole at z_a moves adiabatically around a closed loop C. This calculation is parallel to the one quasihole case:

$$\gamma_C = 2\pi \int_C d^2 z \rho(z) = 2\pi \langle n \rangle \stackrel{!}{=} 2\pi(\frac{\nu}{2\pi\ell^2} S - \frac{1}{q}) \tag{30}$$

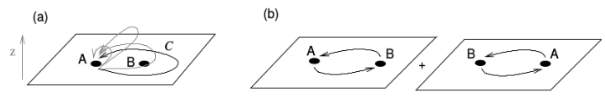

FIG. 4: (a) Process in which a particle A moves on a path C around another particle B. In three space dimensions, one may profit from the third direction (z-direction) to lift the path over particle B and thus to shrink the path into a single point. (b) Process equivalent to moving A on a closed path around B which consists, apart from a topologically irrelevant translation, of two successive exchanges of A and B.

Here the only difference is the density in the contour C is not uniform, but we have another quasihole at z_b, which leads to the density around z_b lower. The reduce of number of electrons is $1/q$. The charge of a quasihole should be independent of the presence of other quasiholes. So we must attribute this extra phase to some other cause. We interpret the extra phase as coming from a fictitious magnetic flux attached to each quasihole. Namely, we consider that the quasiholes are anyons by nature. Therefore, when we treat them as bosons in wavefunction, they appear as composite particles with flux attached to recover their anyonic nature: the exchange or interchange of two quasiholes gives a phase π/q, i.e. the quasi-holes obey fractional statistics.

Fractional Statistics

One of the most exotic consequences of charge fractionalisation in 2D quantum mechanics, exemplified by Laughlin quasi-particles, is fractional statistics. Remember that, in three space dimensions, the quantum-mechanical treatment of two and more particles yields a superselection rule according to which quantum particles are, from a statistical point of view, either bosons or fermions. This superselection rule is no longer valid in 2D (two space dimensions), and one may find intermediate statistics between bosons and fermions.

In order to illustrate the different statistical (i.e. exchange) properties of two quantum particles in three and two space dimensions, consider a particle A that moves adiabatically on a closed path C in the xy-plane around another one B of the same species (see Fig. 4). Path C in the xy-plane around another one B of the same species. We choose the path to be sufficiently far away from particle B and the two particles to be sufficiently localised such that we can neglect corrections due to the overlap between the two corresponding wave functions. Notice first that such a process T is equivalent to two successive exchange processes $T = E^2$.

Let us discuss first the three-dimensional case. Because of the presence of the third direction (z-direction), one may elevate the closed path in this direction while keeping the position of particle A fixed in the xy plane. We have $T(C) = 1$, so $E = 1$ (boson) or $E = -1$ (fermion).

In two space dimensions, this topological argument yields a completely different result. It is not possible to shrink a path C enclosing the second particle B into a single point at the position of A, without passing by B itself. From an algebraic point of view, the exchange processes are no longer described by the two roots of unity, 1 and -1, but by the so-called braiding group. In the simplest case of Abelian statistics,

$$\psi(\mathbf{r}_1)\psi(\mathbf{r}_2) = e^{i\alpha\pi}\psi(\mathbf{r}_2)\psi(\mathbf{r}_1) \tag{31}$$

where α is also called the statistical angle. One has $\alpha = 0$ for bosons and $\alpha = 1$ for fermions, and all other values of α in the interval between 0 and 2 for anyons. Sometimes anyonic statistics is also called fractional statistics - indeed all physical quasi-particles, such as those relevant for the FQHE, have an angle that is a fractional (or rational) number, but there is no fundamental objection that irrational values of the statistical angle should be excluded.

JAIN'S THEORY

In the previous chapter it was demonstrated that the state that causes the fractional quantum Hall effect can be essentially represented by Laughlin's wave function. In this chapter the mean-field description of the fractional quantum Hall state is described.

First of all, we define, a composite fermion is the bound state of an electron and an even number of quantized vortices. The fundamental postulate is, strongly interacting electrons turn into weakly interacting composite fermions, where a composite fermion is a bound state of an electron and an even number of quantized vortices.

First we consider the state at $1/q$ as the starting point of the theory. At this filling factor the real external magnetic field has a strength which corresponds to q magnetic flux quanta per electron. When we replace the electrons by composite fermions, which have $2k$ flux quanta in the opposite direction, the mean field of the fictitious flux cancels part of the external magnetic field such that the effective field for a composite fermion corresponds to $q-2k$ flux quanta per composite fermion. The number $q-2k$ is still odd. Namely, if $q-2k=1$, the effective filling factor ν of the composite fermions is 1, the integer quantum Hall state, and if $q-2k>1$, they are in a fractional quantum Hall state. Therefore, in this picture the fractional quantum Hall states at $\nu=1/q$ are all equivalent to the integer quantum Hall state at $\nu=1$.

Similarly, we can understand the filling $\nu=\frac{n}{2kn+1}$ in the same way, therefore $2/5, 3/7, ...$ can all be understood.

The microscopic meaning of the formation of composite fermions is that, the Hamiltonian becomes

$$H_{MF}=\frac{1}{m_b}(\mathbf{p}_i+e\mathbf{A}^*(\mathbf{r}_i)+e\mathbf{a}(\mathbf{r}_i))^2 \tag{32}$$

where $A*$ produces a uniform magnetic field B^*. We assume that composite fermions are free. The vector potential a^* binds flux quanta to electrons:

$$\mathbf{a}_i^*=\frac{2p\phi_0}{2\pi}\sum_{j\neq i}\frac{\hat{r}_{ij}}{r_{ij}^2}, \quad r_{ij}=|z_i-z_j| \tag{33}$$

which generates a magnetic field

$$\mathbf{b}_i=\partial\times\mathbf{a}(r_i)=2p\phi_0\sum_{j\neq i}\delta^{(2)}(\mathbf{r}_i-\mathbf{r}_j) \tag{34}$$

The corresponding wave function of interacting electrons at ν has the form

$$\Psi^{MF}_{\nu=\frac{n}{2pn\pm1}}(B) = P_{LLL}\Phi_{\pm n}(B^*)\prod_{i<j}(z_i - z_j)^{2p} \quad (35)$$

where $\Phi = \prod_{i<j}(z_i - z_j)\exp[-|z_i|^2/2\ell^2]$ is an antisymmetric wave function for electrons in IQHE. The Jastrow factor, $\prod_{i<j}(z_i - z_k)^{2p}$, binds $2p$ vortices to each electron to convert it into a composite fermion. P_{LLL} denotes projection into the lowest Landau level.

As an example, we now show how the Laughlin wave function can be derived from the CF theory. For the ground state at $\nu = 1/(2p+1)$, the projected wave function reduces to

$$\Psi^{MF}_{\nu=\frac{1}{2p+1}} = P_{LLL}\prod_{i<j}(z_i - z_j)^{2p}\Phi_1, \quad (36)$$

$$\Phi_1 = \prod_{i<j}(z_i - z_j)\exp[-\sum_i |z_i|^2/\ell^2] \quad (37)$$

which gives

$$\Psi^{MF}_{\nu=\frac{1}{2p+1}} = P_{LLL}\prod_{i<j}(z_i - z_j)^{2p+1}\exp[-\sum_i |z_i|^2/\ell^2] \quad (38)$$

This is exactly identical to Laughlin wave function at $\nu = 1/(2p+1)$. (The projection operator does not do anything because the wave function is already in the lowest Landau level.)

How do we know the statistics of a composite fermion? To answer this question, we need to check the exchange operation between two composite fermion: exchange two fermion

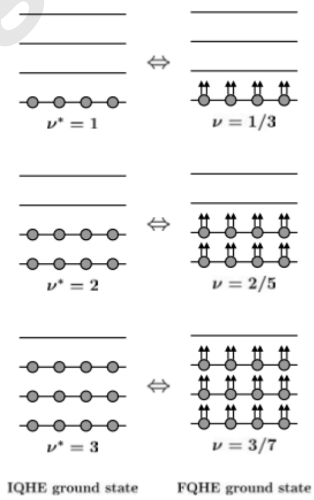

FIG. 5: The composite fermion picture. "*Lambda*" levels are analogous to Landau levels of electrons at B^*.

leads to a factor -1; winding one fermion around a flux quanta by π leads to a factor -1; so that winding a composite fermion (constructed by one electron and two flux quanta) leads to a total factor -1, therefore a composite fermion takes the fermionic statistics.

Furthermore, composite particle theory can be generalized to the composite boson. So that we can predict the FQHE for boson at filling $\nu = 1/2p$ (this is quite different from fermion). Here we can prove a composite particle which encloses one electron and one flux quanta takes bosonic statistics. Then the $\nu = 1/3$ state will be reduced to a bosonic $\nu = 1/2$ state. We can argue bosonic $\nu = 1/2$ state should be a FQHE state.

HALDANE'S THEORY

In this section we show how the FQHE at the filling factors $\nu = p/q$ can be understood from the viewpoint of a hierarchy [F.D.M. Haldane: Phys. Rev. Lett. 51, 605 (1983)].

In this theory, we start our reasoning by noticing that there is a correspondence between the original electron system and the system of quasiparticles. Namely, the electrons have charge and repel each other, which gives rise to a Laughlin state. Similarly, quasiparticles are charged and repel each other. Thus we can expect that they form a Laughlin state at appropriate densities. An important feature is that the quasiparticles do not interact with electrons, since the quasiparticles are excitations from the uniform electron state. We examine the question of at what density the quasiparticles form a Laughlin state.

We consider a system of area S with Ne electrons. In this case the number of single-electron states is $N_s = S/2\pi\ell^2$. Recall that the number of single-particle orbital is related to maximum power of the Laughlin wave function: $N_s = M + 1$. So we know that, for a single quasihole wave function $\prod_i (z_i - z_b)\Psi_q$, it is a polynomial of order N_e (for z_a), so the number of single-quasihole states is $N_s^{q.h.} = N_e + 1$. If we consider a multi-quasi-hole state:

$$\prod_i (z_i - z_a) \prod_j (z_j - z_b) \prod_i (z_k - z_c) ... \times \Psi_q \tag{39}$$

Here $z_a, z_b...$ indicate the positions of the quasiholes. The orders of the polynomial with respect to each quasi-hole is still N_e, so the number of single quasi-hole states is $N_s^{q.h.} = N_e + 1$. Here we assume the quasiholes are boson, or they are not the identical particles. Bosonic Laughlin state should be written as $\prod (z_i - z_j)^{2p}$ at filling $\nu = 1/2p$, by directly extending the fermioic Laughlin state. Namely, the state is realized when $\nu^{q.h.} = 1/2p = N_e^{q.h.}/N_s^{q.h.}$,

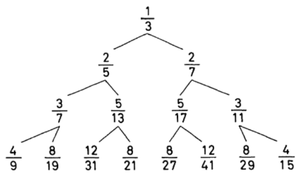

FIG. 6: Haldane hierarchy state by setting $p_i = 1$ based on Laughlin $\nu = 1/q = 1/3$.

where $N_e^{q.h.}$ is the number of quasi-holes. Please distinguish the filling factor of quasi-holes $\nu^{q.h.}$ from that of electrons ν. Finally, we calculate the filling factor of electrons to fill the consistent relationship. Say, the largest power of an electron in the multiple quasi-hole wave function is $(N_e - 1)q + N_e^{q.h.}$, which should be equal to $N_s - 1$. The filling of electrons therefore is

$$\nu = \frac{N_e}{N_s} = \frac{N_e}{(N_e - 1)q + N_e^{q.h.}} = \frac{N_e}{(N_e - 1)q + (N_e + 1)/2p} \sim \frac{2p}{2pq + 1}. \tag{40}$$

As an example, we put $p = 1$ to construct the state at $\nu = 2/5$. What we have shown at this point is that if quasiparticles are introduced into the Laughlin state of electrons, which we call the parent generation, the quasi particles, which belong to the daughter generation, form a Laughlin state of their own. We can take this relation further: when quasiparticles of the granddaughter generation are introduced into the daughter generation Laughlin state of quasiparticles, they form a Laughlin state at appropriate densities. Similarly, a great-granddaughter generation Laughlin state will be formed, and so on. As a result, the filling factor of electrons when the Laughlin state is formed at some stage of these generations is as follows:

$$\nu = \frac{1}{q + \frac{\alpha_1}{2p_1 + \frac{\alpha_2}{2p_2 + \ldots}}} \tag{41}$$

$\alpha = 0, \pm 1$, and p_i is an arbitary positive integer.

CONFORMAL FIELD THEORY

Certain FQHE wave functions can be connected to certain correlation functions of chiral conformal field theory (Cristofano Coulomb gas approach to quantum Hall effect. Phys. Lett. B262, 88 (1991); Fubini Vertex operators and quantum Hall effect. Mod. Phys. Lett. A 6, 347 (1991); Moore and Read 1991 Nonabelions in the fractional quantum Hall effect. Nucl. Phys. B 360, 362 (1991).), which is very briefly outlined in this section. We use some standard results from conformal field theory without derivation. (The derivations can be found, for example, in the textbook of Di Francesco, Mathieu, and Senechal).

Consider a free bosonic field in 1+1-dimensional Euclidean spacetime, with its correlator given by

$$\langle \phi(z)\phi(z')\rangle = -\ln(z-z') \tag{42}$$

The so-called vertex operators are defined by

$$V_\alpha(z) = e^{i\alpha\phi(z)} \tag{43}$$

With the help of Wicks theorem, their correlators can be shown to be given by the expression

$$\langle \prod_i V_{\alpha_i}(z_i)\rangle \exp[-\sum_{i<j} \alpha_i\alpha_j \langle \phi(z_i)\phi(z_j)\rangle] = \prod_{i<j}(z_i - z_j)^{\alpha_i\alpha_j} \tag{44}$$

Disregarding the neutrality condition, the choice $\alpha_i = q$ gives precisely the Laughlin wave function on the right hand side. The gaussian part can be obtained by the neutrality condition.

There is no fundamental reason why the correlation functions (or conformal blocks) of vertex operators in a two-dimensional Euclidean conformal field theory should bear any relation to the quantum mechanical wave functions of electrons in the lowest Landau level interacting via the Coulomb potential. Nonetheless, one can ask if some other correlation functions in conformal field theory may also qualify as legitimate FQHE wave functions. Any ansatz wave function must be tested and confirmed in a quantum mechanical calculation.

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[1] E. Fradkin, Field Theories of Condensed Matter Physics. Cambridge University Press, 2013.