

**Lecture notes for**  
**The Integer Quantum Hall Effect and Beyond**

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## INTRODUCTION

This course will address the topics related to the quantum Hall effect. Personally I think it is a very important subject, which should be an introductory level course to graduate students in Condensed Matter Physics. The reason is many-fold. First of all, the quantum Hall physics is one of two most important discoveries in physics, since 1980s. The concepts, ideas and knowledges reflected in this area totally reshaped the modern physics especially condensed matter physics. It changes the way that people think about problems. Moreover, the discovery of quantum Hall effect gives birth to the topological physics. Nowadays, topology is one of most exciting field in condensed matter physics. Without any doubt, the physics related to topology is one of most exciting field in condensed matter physics since 1980s. There are in total three Nobel prizes related to this topic: the discovery of integer quantum Hall effect (1980, Nobel prize in 1985), the discovery of fractional quantum Hall effect (1982, Nobel prize in 1998), the topological states of matter (1974, 1982, 1984, 1988, Nobel prize in 2016). We will try to cover these progresses in this lecture.

The basic knowledges in the quantum Hall effect are important to understand where the modern condensed matter physics comes from, why condensed matter physics becomes “condensed matter”, and how it looks like now. So these knowledges should be the ground for a graduate student of condensed matter physics.

Topological states of matter is a surprising outcome of condensed matter physics, where systems that are full of details give rise to physical phenomena that are astonishingly independent of many irrelevant setup details. The most striking example, the quantum Hall effect, shows a quantization of the Hall resistivity to a level within error around  $10^{-9}$ , with the quantum being as universal as physics gets,  $h/e^2$ . Soon afterwards, the fractional quantum Hall effect exposed the way in which interaction between electrons may lead to wide spread fractionalization - of the charge, the spin, the statistics, the central charge, and of several response functions. In 2005, the discovery of the time-reversal-symmetric analogs of the quantum Hall effect, the topological insulators, comes into the condensed matter physics.

Nowadays, topological states of matter has been spaned to a wide family. Here we just mention a general framework. A naive definition of topological states of matter: Some of physical properties are robust due to intrinsic reasons (e.g. symmetries), and usually these properties can be described by some topological quantum numbers, which are independent

of details of setup. Many of interested states can be grouped into the following table or graph.

TABLE I: A general classification of topological states of matter

Topological States of Matter	
Topological Band Structure	Intrinsic Topological Order
Integer quantum Hall Effect	Fractional Quantum Hall Effect
Quantum Anomalous Hall Effect (Chern Insulator)	Fractional Chern Insulator
Quantum Spin Hall Effect	Quantum Spin Liquid
Topological Insulator	Fractional Topological Insulator
Topological Crystalline Insulator	
Topological Semimetal	

The quantum Hall physics, in the center stage of topological states of matter, has become an extremely important research subject during the last two and a half decades. The interest for quantum Hall physics stems from the low-dimensional quantum systems to strongly-correlated systems, and probably covers many key areas of modern condensed-matter physics. From a theoretical point of view, the study of quantum Hall systems brings many novel concepts some of which were better known in quantum-field theories used in high-energy, such e.g. charge fractionalisation, non-commutative geometries and topological field theories. And one will see many exotic techniques and tools for topological states that is beyond the standard solid state physics. Thus, I choose it as the main focus of this lecture.

The motivation of the present lecture notes is to provide in an accessible manner the basic knowledge of the quantum Hall physics and topological states of matter. I hope it will let interested graduate students to pursue on their own further studies in this subject. I feel that a more detailed discussion of some aspects in this large field of physics would go beyond the introductory character of this lecture note. Finally, through this lecture, I hope to clarify and answer the following questions:

- What do we mean by “Insulators”?

- What do we mean by “Topology” ?
- What do we mean by “Topological Insulators” ?
- What do we mean by “Fractional” ?

I hope these questions will be clear after this lecture is done.

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## PRELIMINARY

### Electric and magnetic field

These are two most common field in physics, which are widely studied in the past centuries. An electric field is generated by a voltage, and a magnetic field is generated by magnet. Their relationship can be clearly explained with the help of Maxwell's Equations, a set of partial differential equations which relate the electric and magnetic fields to their sources, current density and charge density.

$$\text{Gauss' slaw : } \oint_S \mathbf{E} \cdot d\mathbf{s} = Q \quad (1)$$

$$\text{Gauss' slaw formagnet : } \oint_S \mathbf{B} \cdot d\mathbf{s} = 0 \quad (2)$$

$$\text{Faraday' slaw : } \oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{s} \quad (3)$$

$$\text{Ampre - Maxwell' slaw : } \oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 [I_0 + \frac{d}{dt} \int_S \mathbf{E} \cdot d\mathbf{s}] \quad (4)$$

**Gauss's law:** The electric flux through any closed surface is equal to the electric charge  $Q$  enclosed by the surface. Gauss's law describes the relation between an electric charge and the electric field it produces. This is often pictured in terms of electric field lines originating from positive charges and terminating on negative charges, and indicating the direction of the electric field at each point in space.

**Gauss's law for magnetism:** The magnetic field flux through any closed surface is zero. This is equivalent to the statement that magnetic field lines are continuous, having no beginning or end. Any magnetic field line entering the region enclosed by the surface must also leave it. No magnetic monopoles, where magnetic field lines would terminate, are known to exist

**Faraday's law:** A changing magnetic field induces an electromotive force and, hence, an electric field. The direction of the force opposes the change. Changing magnetic flux through a surface induces an electromotive force (EMF) in any boundary path of that surface. A changing magnetic field induces a circulating electric field. The voltage accumulated around a closed circuit is proportional to the time rate of change of the magnetic flux it encloses.

**Ampre's law:** An electric current  $I$  or a changing electric flux through a surface produces a circulating magnetic field around any path that bounds that surface. Electric currents

and changes in electric fields are proportional to the magnetic fields circulating about the areas where they accumulate.

We can also express the Maxwell equations in the differentiation form:

$$\partial \cdot \mathbf{E} = \rho \quad (5)$$

$$\partial \cdot \mathbf{B} = 0 \quad (6)$$

$$\partial \times \mathbf{E} = -\frac{d}{dt}\mathbf{B} \quad (7)$$

$$\partial \times \mathbf{B} = \mu_0[\mathbf{J} + \frac{d}{dt}\mathbf{E}] \quad (8)$$

The usual gauge choice is like following.  $\mathbf{E} = -\partial\phi - \frac{\partial\mathbf{A}}{\partial t}$  and  $\mathbf{B} = \partial \times \mathbf{A}$ . This choice satisfies the Maxwell equation.

When an electron moving in a electric and magnetic field, the Lorentz force is

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \quad (9)$$

In an electromagnetic field, the directions in which the electric and magnetic field move, are perpendicular to each other.

### The classical Hall effect

The behaviour of charged particles moving in a magnetic field has been of interest to physicists for many decades. In 1879, E. H. Hall carried out the first measurements of the current flowing through a thin metallic strip in the presence of a perpendicular magnetic field, using the set-up shown in Fig. 1. Remarkably, he detected a current that flows in a perpendicular direction to the applied voltage, and whose strength is proportional to the applied magnetic field and the longitudinal current.

This behaviour was attributed to the Lorentz force,  $F = q[E + v \times B]$ , which acts on a particle with charge  $q$  travelling with velocity  $v$  in the presence of an electric field,  $E$ , and magnetic field,  $B$ . In free (two-dimensional) space, the presence of a perpendicular magnetic field causes moving charged particles to form circular cyclotron orbits, oscillating at the cyclotron frequency,  $\omega_c = |q|B/m$ . [Classically, because of the electrons move always perpendicular to the magnetic field the Lorentz force can be simplified to  $F = ev_0B$ . You can get the radius because the Lorentz force is here the centripetal force  $F = \frac{mv_0^2}{r}$ , with the

electron mass  $m$  and the radius  $r$ . Then,  $r = \frac{mv_0}{eB}$ . So the period is  $T = \frac{2\pi r}{v_0} = 2\pi \frac{m}{eB}$ .] In a classical two-dimensional slab of metal, however, deflected electrons near the edge cannot complete circular orbits, and so charge starts to build up at the side of the sample. In turn, this creates a transverse voltage and electric field, which eventually opposes the Lorentz force. When these forces are balanced, the equilibrium potential difference is known as the Hall voltage ( $V_H$ ), and leads to a current in a four-terminal measurement.

The classical Hall effect refers to the phenomenon whereby a current carrying conductor (with current  $I$ ) exposed to an external magnetic field  $\mathbf{B}$ , develops a transverse potential difference  $V_H$  (and conversely). We define the Hall resistance as

$$R_H = \frac{V_H}{I} = \frac{b \cdot \mathbf{E}}{b \cdot \mathbf{j}} = \frac{vB}{qn_e v} = \frac{B}{qn_e} \quad (10)$$

where  $q$  is the charge of the current carriers and  $n_e$  is the number of carriers per unit area.

Remark. 1.  $R_H$  is large when  $n_e$  is small, so that thin conductors generate large Hall resistance.

2. By measuring  $R_H$  we can determine the sign of  $q$  and thus determine whether current is carried by electrons or holes.

More quantitatively, the classical Hall effect may be understood within the Drude model for diffusive transport in a metal. In this model, we consider independent charge carriers of momentum  $\mathbf{p}$  described by the equation of motion

$$\frac{d\mathbf{p}}{dt} = -e(\mathbf{E} + \frac{\mathbf{p}}{m} \times \mathbf{B}) - \frac{\mathbf{p}}{\tau} \quad (11)$$

The last term takes into account relaxation processes due to the diffusion of electrons by generic impurities, with a characteristic relaxation time  $\tau$ . The macroscopic transport characteristics, i.e. the resistivity or conductivity of the system, are obtained from the static solution of the equation of motion,  $dp/dt = 0$ , and one finds

$$eE_x = -\frac{eB}{m}p_y - p_x/\tau \quad (12)$$

$$eE_y = \frac{eB}{m}p_x - p_y/\tau \quad (13)$$

where we have assumed the magnetic field is along z-direction. The cyclotron frequency is  $\omega_c = \frac{eB}{m}$ . With the help of the Drude conductivity,  $\sigma_0 = \frac{ne^2\tau}{m}$ , we have

$$\sigma_0 E_x = -enp_x/m - enp_y/m(\omega_c\tau) \quad (14)$$

$$\sigma_0 E_y = enp_x/m(\omega_c\tau) - enp_y/m \quad (15)$$

or in terms of the current density  $\mathbf{j} = -en\mathbf{p}/m$ , and in the matrix form as  $\mathbf{E} = \boldsymbol{\rho}\mathbf{j}$ , the resistivity tensor is

$$\boldsymbol{\rho} = \boldsymbol{\sigma}^{-1} = \frac{1}{\sigma_0} \begin{bmatrix} 1 & \omega_c\tau \\ -\omega_c\tau & 1 \end{bmatrix} = \frac{1}{\sigma_0} \begin{bmatrix} 1 & -\mu B \\ -\mu B & 1 \end{bmatrix} \quad (16)$$

and the mobility is  $\mu = \frac{e\tau}{m}$ . From the above expression, one may immediately read off the Hall resistivity  $\rho_H = \frac{\omega_c\tau}{\sigma_0} = \frac{B}{en}$ . Furthermore, we have conductivity tensor is

$$\boldsymbol{\sigma} = \boldsymbol{\rho}^{-1} = \sigma_0 \begin{bmatrix} 1/(1 + \omega_c^2\tau^2) & -\omega_c\tau/(1 + \omega_c^2\tau^2) \\ \omega_c\tau/(1 + \omega_c^2\tau^2) & 1/(1 + \omega_c^2\tau^2) \end{bmatrix} \quad (17)$$

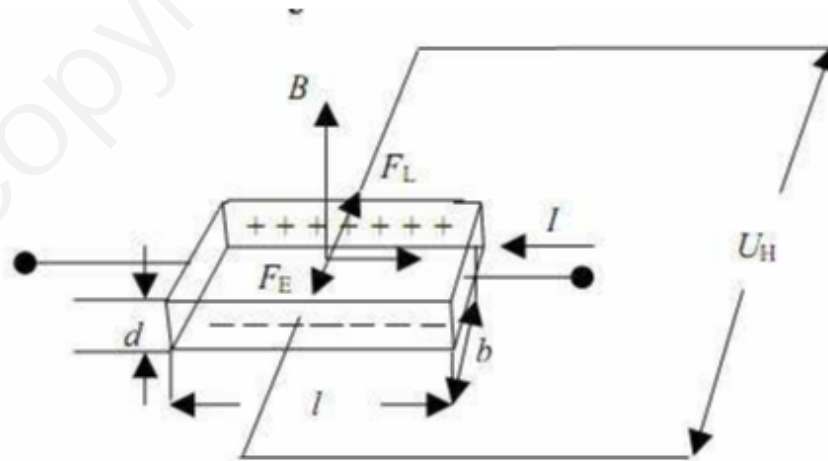


FIG. 1: The classical Hall Effect.

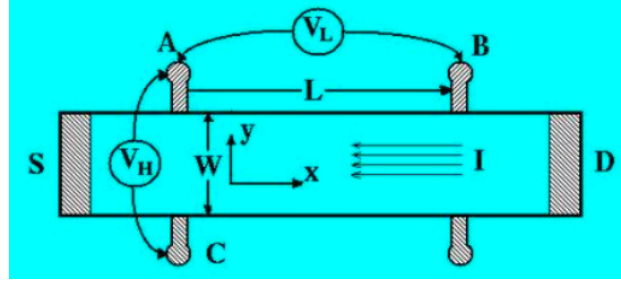


FIG. 2: The Hall bar setup.

## EXPERIMENTAL OBSERVATIONS

### EXPERIMENTAL SETUP

#### Hall bar

The Hall bar setup is shown in Fig. 2. The experimental sample is placed in the center. A current  $I$  is sent through the material, along the direction labeled  $x$ . The voltage change  $V_x$  is measured between the two leads on top. This configuration is a standard four-probe measurement. The voltage contacts are not put at the end of the sample, in order to avoid the influence of nonohmic contacts where the current enters and leaves. Another pair of voltage contacts measure the voltage  $V_y$  across the sample. This voltage is zero in the absence of a magnetic field.

#### Resistance and resistivity

We have the relation:

$$\begin{bmatrix} E_1 \\ E_2 \end{bmatrix} = \begin{bmatrix} \rho_{11} & \rho_{21} \\ \rho_{12} & \rho_{22} \end{bmatrix} \begin{bmatrix} j_1 \\ j_2 \end{bmatrix} = \begin{bmatrix} \rho_{11} & \rho_{21} \\ \rho_{12} & \rho_{22} \end{bmatrix} \begin{bmatrix} I/W \\ 0 \end{bmatrix} = \frac{I}{W} \begin{bmatrix} \rho_{11} \\ \rho_{21} \end{bmatrix} \quad (18)$$

So we have the longitudinal voltage is

$$V_L = E_1 L = \rho_{11} \frac{L}{d} I, \quad R_L = \rho_{11} \frac{L}{d} \quad (19)$$

and the transverse voltage is

$$V_H = E_2 W = \rho_{21} \frac{I}{W} W = \rho_{21} I, \quad R_H = \rho_{21} \quad (20)$$

In two dimensional Hall bar, we have that the resistance  $R_H$  is equal to the resistivity  $\rho$ . This relation is important for experiments.

### Two-dimensional electron gas

As already mentioned above, the history of the quantum Hall effect is related to technological advances in the fabrication of 2D electron systems with high electronic mobilities. The increasing mobility allows one to probe the fine structure of the Hall curve and thus to observe those quantum Hall states which are more fragile, such as some exotic FQHE states. In this sense, electronic mobility means resolution and the tiny object is the quantum Hall state. As an order of magnitude, today's best 2D electron gases (in GaAs/AlGaAs heterostructures) are characterised by mobilities  $\mu \sim 10^7 \text{cm}^2/\text{Vs}$ .

Remark. 1) high mobility; 2) 2D nature.

### MOSFET

The samples used in the discovery and in the first studies of the IQHE were metal-oxide-semiconductor field-effect transistors (MOSFET). A metallic layer is separated from a semiconductor (typically doped silicon) by an insulating oxide (e.g. SiO<sub>2</sub>) layer (see inset I in Fig. ). The chemical potential in the metallic layer may be varied with the help of a gate voltage  $V_G$ . At  $V_G = 0$ , the Fermi energy in the semiconductor lies in the band gap below the acceptor levels of the dopants [Fig. 3(a)]. When lowering the chemical potential in the metal with the help of a positive gate voltage  $V_G > 0$ , one introduces holes in the metal that attract, via the electric field effect, electrons from the semiconductor to the semiconductor-insulator interface. These electrons populate the acceptor levels, and as a consequence, the semiconductor bands are bent downwards when they approach the interface, such that the filled acceptor levels lie now below the Fermi energy (Fig. 3(b)). Above a certain threshold of the gate voltage, the bending of the semiconductor bands becomes so strong that not only the acceptor levels are below the Fermi energy, but also the conduction band in the vicinity of the interface which consequently gets filled with electrons [Fig.3 (c)]. One thus obtains the electrons in the conduction band, the dynamics of which is quantised into discrete electronic subbands in the perpendicular z-direction (see inset II in Fig. 3). Naturally, the electronic

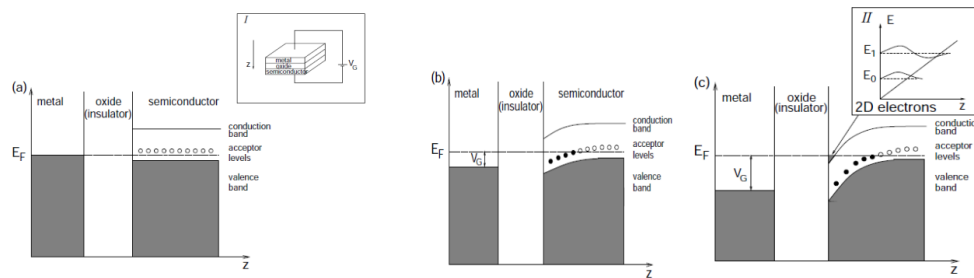


FIG. 3: The inset I shows a sketch of a MOSFET. (a) Level structure at  $V_G = 0$ . In the metallic part, the band is filled up to the Fermi energy  $E_F$  whereas the oxide is insulating. In the semiconductor, the Fermi energy lies in the band gap (energy gap between the valence and the conduction bands). Close to the valence band, albeit above  $E_F$ , are the acceptor levels. (b) The chemical potential in the metallic part may be controlled by the gate voltage  $V_G$  via the electric field effect. As a consequence of the introduction of holes the semiconductor bands are bent downwards, and above a threshold voltage (c), the conduction band is filled in the vicinity of the interface with the insulator. One thus obtains a 2D electron gas. Its levels (electronic subbands) are represented in the inset II.

wave functions are then extended in the  $z$ -direction, but in typical MOSFETs only the lowest electronic subband is filled, such that the electrons are purely 2D from a dynamical point of view, i.e. there is no electronic motion in the  $z$ -direction.

### *Semiconductor heterostructures*

The mobility in MOSFETs, which is typically on the order of  $\mu \sim 10^6 \text{ cm}^2/\text{Vs}$ , is limited by the quality of the oxide-semiconductor interface (surface roughness). This technical difficulty is circumvented in semiconductor heterostructures – most popular are GaAs/AlGaAs heterostructures – which are grown by molecular-beam epitaxy (MBE), where high-quality interfaces with almost atomic precision may be achieved, with mobilities on the order of  $\mu \sim 10^7 \text{ cm}^2/\text{Vs}$ .

We put a junction of AlGaAs (with ratio 4 : 1 for the Aluminium) and GaAs together. The electrons are bound to the interface so we obtain a two-dimensional electron gas (2DEG). The gas is bound on the interface but free along it. To understand the interface better, we first picture the two materials separately. The materials are set up in such a way that the

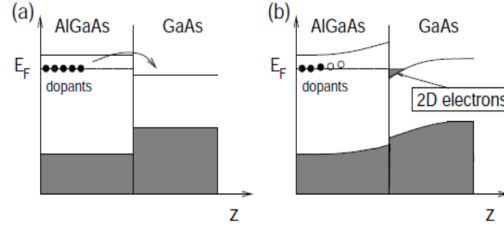


FIG. 4: Semiconductor heterostructure (GaAs/AlGaAs). (a) Dopants are introduced in the AlGaAs layer at a certain distance from the interface. The Fermi energy lies below the in the band gap and is pinned by the dopant levels. The GaAs conduction band has an energy that is lower than that of the dopant levels, such that it is energetically favourable for the electrons in the dopant layer to populate the GaAs conduction band in the vicinity of the interface. (b) This polarisation bends the bands in the vicinity of the interface between the two semiconductors, and thus a 2D electron gas is formed there on the GaAs side.

Fermi energy of AlGaAs is higher than that of GaAs. At zero temperature the Fermi level on the  $\text{Al}_x\text{Ga}_{1-x}\text{As}$  side lies just above the bound state of the donors, which lies higher than the bottom of the GaAs conduction band. (The Fermi energy is pinned by these donor levels in AlGaAs, which may have a higher energy than the originally unoccupied conduction band in the GaAs part.) Therefore, electrons bound to the donors move into the GaAs conduction band. (it becomes energetically favourable for the electrons in the donor levels to occupy the GaAs conduction band in the vicinity of the interface. As a consequence, the energy bands of AlGaAs are bent upwards, whereas those of GaAs are bent downwards.) This replacement causes electric polarization: the GaAs is charged negatively and the  $\text{Al}_x\text{Ga}_{1-x}\text{As}$  is charged positively. So when we bring the two media into contact, electrons (assumed with positive charge) spill over from donor sites of AlGaAs, which creates a dipole layer and potential difference. In this way, two-dimensional electrons are formed automatically in this system.

The typical 2D electronic densities in these systems are on the order of  $n_e \sim 10^{11}\text{cm}^{-2}$ , which is much lower than in usual metals. The mobilities is on the order of  $\mu \sim 10^7\text{cm}^2/(\text{Vs})$ . The high mobility is necessary to observe the FQHE, which was indeed first observed in a GaAs/AlGaAs sample.

Remark: Other 2D electron gas for quantum Hall systems: Graphene; Si/SiGe; GaAlAs/AlAs/GaAlAs; ZnO/MgZnO:

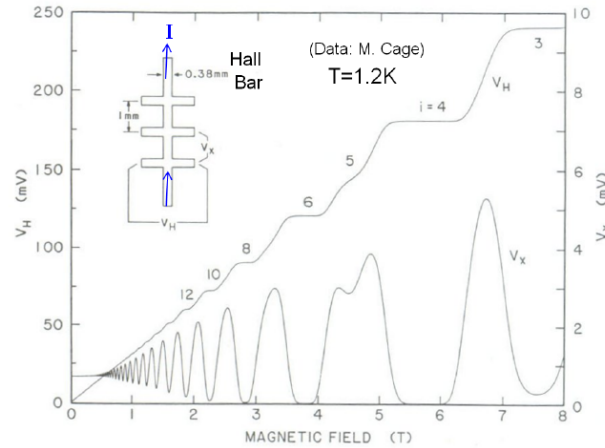


FIG. 5: The experimental observation of IQHE.

### Experimental Results

The integer quantum Hall effect (IQHE) is the experimental discovery (von Klitzing, 1980) that the Hall resistance takes integer values.

The experiment exhibits the following features: 1. Plateaus of Hall resistance takes quantization at some interger values.

2. For integer or near integer values, the longitudinal resistance is zero.

3. The width of the plateaus increases with disorder. In fact the disorder will turn out to be crucial for the quantization effect, though too much disorder will destroy it.

4. The accuracy os the quantization is  $10^{-8}$ . Indeed, the value of  $h/e^2$  has been measured and found to be  $h/e^2 = 25812.807572\Omega$ .

Remark. 1. Low temperature 2. High magnetic field.

## THE LANDAU LEVEL PROBLEM

*Tips: Landau level. 1929 by Lev Landau*

The appearance of the Planck constant  $\hbar$  in the formula for  $R_H$  is an indication of the inherently quantum mechanical nature of the effect. In this chapter, we study a single electron confined to two dimensions and exposed to a magnetic field. This problem was solved exactly soon after the invention of quantum mechanics (Landau 1929), because it is merely a one-dimensional simple harmonic oscillator problem. The most remarkable aspect of the solution is that the electron kinetic energy is quantized. The discrete kinetic energy levels are called “Landau levels”.

### Hamiltonian

Our system is free spinless electrons subject to a magnetic field perpendicular to the two dimensional plane. We work in the one-electron approximation, in the one-particle Hilbert space. The Hamiltonian (due to Landau) is given by

$$H = \frac{1}{2m}(\mathbf{p} - e\mathbf{A})^2 \quad (21)$$

where  $\mathbf{A}(\mathbf{r})$  is the vector potential that generates the magnetic field

$$\mathbf{B} = \nabla \times \mathbf{A} \quad (22)$$

In this equation  $p$  is the canonical momentum operator, which satisfies the canonical commutation relation  $[p_a, r_b] = -i\hbar\delta_{ab}$ . The substitution is also called minimal substitution.

Just as in classical mechanics, the canonical momentum operator is not proportional to the velocity operator. The latter,  $v$ , is derived from the Heisenberg equation of motion as follows:  $v_a = i[H, r_a] = (\mathbf{p} - e\mathbf{A})_a/m$ . We introduce what we call the dynamical momentum operator  $\pi_a$ , which is proportional to the velocity operator:  $\pi_a = mv_a = (\mathbf{p} - e\mathbf{A})_a$ .

The momentum operator satisfies

$$[\pi_a, \pi_b] = i\hbar eB = -i\frac{\hbar^2}{\ell^2} \quad (23)$$

where the magnetic length  $\ell = \sqrt{\hbar/(|e|B)}$ . Let us prove it here.

$$\begin{aligned} [\pi_x, \pi_y] &= [\mathbf{p}_x - e\mathbf{A}_x(\mathbf{r}), \mathbf{p}_y - e\mathbf{A}_y(\mathbf{r})] \\ &= e([p_x, A_y] - [p_y, A_x]) = e\left(\frac{\partial A_x}{\partial x}[p_x, x] + \frac{\partial A_x}{\partial y}[p_x, y] - \frac{\partial A_y}{\partial x}[p_y, x] - \frac{\partial A_y}{\partial y}[p_y, y]\right) \\ &= i\hbar e\left(\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x}\right) = i\hbar eB \end{aligned}$$

This momentum operator also does not commute with the Hamiltonian. There is big physics behind! This is similar to the Heisenberg relation in spin system, which involves non-commutative relation. This is a fully Quantum result, making it quantum! This non-commutativity between the position and its associated momentum is the origin of the Heisenberg inequality according to which one cannot know precisely both the position of a quantum mechanical particle.

The magnetic length is the fundamental length scale in the presence of a magnetic field. For a estimation,  $\ell \approx 26nm/\sqrt{B[T]}$ . ( $T = Vs/m^2 = J/(Am^2)$ ,  $\hbar \approx 10^{-34} Js$ ,  $e = 1.6 \times 10^{-19} As$ )

### Gauge invariance

The choice of the vector potential is not unique for a given magnetic field. This is called a “freedom in the gauge”. Now, the Hamiltonian does not depend on the magnetic field directly but on the vector potential. Thus different choices of the vector potential give different results for the wave function. We need to know how the wave function depends on the choice of the vector potential.

For an arbitrary smooth, real function  $\xi(r)$ ,  $A'(r)$  obtained from  $A'(r) = A(r) + \partial\xi(r)$  and  $A(r)$  stand for the same magnetic field (no singularity in  $\xi(r)$ ). Under such gauge transformation, the wave function should be changed to  $\psi(r) \rightarrow e^{-i\frac{e}{\hbar}\xi(r)}\psi(r)$ .

Notice that because  $A(r)$  is not gauge invariant, neither is the momentum  $p$ . Indeed, the momentum transforms as  $p_a \rightarrow p_a - e\nabla_a\xi(r)$  under a gauge transformation in order to compensate the transformed vector potential, such that  $\pi_a$  is gauge-invariant.

### Eigenvalue problem

For solving the Hamiltonians, it is convenient to use the pair of conjugate operators  $\pi_x$  and  $\pi_y$  to introduce ladder operators in the same manner as in the quantum-mechanical treatment of the one-dimensional harmonic oscillator:  $a = \frac{\ell}{\sqrt{2\hbar}}(\pi_x - i\pi_y)$ ,  $a^\dagger = \frac{\ell}{\sqrt{2\hbar}}(\pi_x + i\pi_y)$ , which satisfies  $[a, a^\dagger] = 1$ , the Hamiltonian is rewritten as

$$H = \hbar\omega_B(a^\dagger a + \frac{1}{2}) \quad (24)$$

Thus the energy eigenvalues are discretized into  $E_n = \hbar(n + 1/2)$  like a harmonic oscillator. These discrete energy levels are called the ‘‘Landau levels’’. The Landau levels are obtained from the commutation relation of  $\pi_a$ , so they do not depend on the choice of the gauge; this is as it should be.

The number operator  $a^\dagger a|n\rangle = n|n\rangle$ . The ladder operators act on these states in the usual manner  $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$  and  $a|n\rangle = \sqrt{n}|n-1\rangle$ .

Next we will work out the eigenvalue problem explicitly.

1) We select the Landau gauge

$$\mathbf{A} = B(0, x, 0) \quad (25)$$

the Hamiltonian does not contain  $y$ , and therefore commutes with  $p_y$ . That implies that  $p_y = \hbar k_y$  is a good quantum number:

$$\frac{1}{2m}[-\partial_x^2 + (p_y - eBx)^2]\psi(x, y) = E\psi(x, y) \quad (26)$$

$$\frac{1}{2m}[-\partial_x^2 + (\hbar k_y - eBx)^2]\psi(x) = E\psi(x) \rightarrow \frac{1}{2m}[-\partial_x^2 + \frac{1}{\ell^2}(\ell k_y - x/\ell)^2]\psi(x) = E\psi(x) \quad (27)$$

where we set the form of wave function as  $\psi(x, y) = e^{ik_y y}\psi(x)$ .

In terms of dimensionless quantities

$$x' = x/\ell - \ell k_y \quad (28)$$

$$p'_x = \ell p_x/\hbar \quad (29)$$

we have the hamiltonian as

$$H = \frac{\hbar\omega_B}{2}[(x')^2 + (p'_x)^2], [-\partial_x^2 + x^2]\psi(x) = E/\hbar\omega_B\psi(x) \quad (30)$$

which is the familiar Hamiltonian of a one-dimensional harmonic oscillator. Here,  $\hbar\omega_B$  is the cyclotron energy, with  $\omega_B = \frac{eB}{m}$ .

The eigenvectors are

$$\psi_{n,k_y}(r) = \frac{1}{[\pi 2^{2n} (n!)^2]^{1/4}} \exp[ik_y y] \exp[-\frac{(x - k_y \ell^2)^2}{2\ell^2}] H_n\left(\frac{x - k_y \ell^2}{\ell}\right) \quad (31)$$

Here we emphasize two points: 1) The energy does not depend on  $k_y$ . The eigenstates with different  $k_y$  in a given Landau level are degenerate. 2) The x position depends on  $k_y$ . An eigenfunction is Gaussian-localized in a narrow strip of width  $\sim \ell$  centered at  $x_m = k_y \ell^2$ . 3) To solve the differential equation for the harmonic oscillator  $\frac{\partial^2 \psi(x,y)}{\partial^2 x} + (E - x^2)\psi(x,y) = 0$ , first notice  $k_y$  is a good quantum number, so we use the form for wave function  $\psi(x,y) = e^{ik_y y} \psi(x)$ . At the second step, we need consider the boundary condition  $x \rightarrow \infty$  we have  $\psi(x) \rightarrow 0$ , so the form of wave function should be  $\psi(x) \sim u(x)e^{-x^2/2}$ . The solution of we get  $\partial^2 u(x)/\partial^2 x - 2x\partial u(x)/\partial x + (E - 1)u(x) = 0$  is  $u(x) = H(x)$ .

2) The symmetric gauge:  $\mathbf{A} = B(-y/2, x/2, 0)$ .

$$H = \frac{1}{2} \left[ (-i\partial_x - \frac{y}{2})^2 + (-i\partial_y + \frac{x}{2})^2 \right] = \frac{1}{2} \left[ -4 \frac{\partial^2}{\partial z \partial \bar{z}} + \frac{1}{4} z \bar{z} - z \frac{\partial}{\partial z} + \bar{z} \frac{\partial}{\partial \bar{z}} \right] \quad (32)$$

where we define  $z = x - iy = re^{-i\theta}$ ,  $\bar{z} = x + iy = re^{i\theta}$ ,  $\partial_x = \partial_z + \partial_{\bar{z}}$ ,  $\partial_y = -i(\partial_z - \partial_{\bar{z}})$ . The ladder operators can be defined as

$$a^\dagger = \frac{1}{\sqrt{2}} \left( \frac{\bar{z}}{2} - 2 \frac{\partial}{\partial \bar{z}} \right) \quad (33)$$

$$a = \frac{1}{\sqrt{2}} \left( \frac{z}{2} + 2 \frac{\partial}{\partial z} \right) \quad (34)$$

$$b^\dagger = \frac{1}{\sqrt{2}} \left( \frac{z}{2} - 2 \frac{\partial}{\partial \bar{z}} \right) \quad (35)$$

$$b = \frac{1}{\sqrt{2}} \left( \frac{\bar{z}}{2} + 2 \frac{\partial}{\partial z} \right) \quad (36)$$

$$[a, a^\dagger] = 1, \quad [b, b^\dagger] = 1 \quad (37)$$

The hamiltonian becomes

$$H = a^\dagger a + 1/2 \quad (38)$$

In addition, The z component of the angular momentum operator is defined as

$$L_z = -i\hbar \frac{\partial}{\partial \theta} = -\hbar \left( z \frac{\partial}{\partial z} - \bar{z} \frac{\partial}{\partial \bar{z}} \right) = -\hbar (b^\dagger b - a^\dagger a) \quad (39)$$

Exploiting the property  $[H, L_z] = 0$ , the eigenfunctions are chosen to diagonalize H and L simultaneously. The eigenvalue of L is denoted by  $-m\hbar$ ; with this definition the quantum number m takes values  $-n, -n + 1, \dots$

The ground state wave function is solved by  $a|0, 0\rangle = 0, b|0, 0\rangle = 0$ . We obtain

$$\langle r|0, 0\rangle = \frac{1}{\sqrt{2\pi\ell}} e^{-\frac{|z|^2}{4}}. \quad (40)$$

The other wavefunctions are obtained by (The wave function in the lowest Landau level)

$$\langle r|n, m\rangle = \frac{(a^\dagger)^n (b^\dagger)^m}{\sqrt{n!} \sqrt{m!}} \langle r|0, 0\rangle \quad (41)$$

This wave function represents an electron localized circularly as shown in disk. The maximum of the existence probability is on the circumference of a circle of radius  $\sqrt{2m}\ell$ , and the spread of the wave function in the radial direction is of the order of  $\ell$ . The expectation value of  $r^2$  is  $\langle 0, m|r^2|0, m\rangle = 2(m+1)\ell^2$ .

Exercises. Prove the normalization condition of electron wave functions.

$$\langle 0, m|0, m'\rangle = \int \frac{1}{\sqrt{2\pi 2^m m!} \ell} z^m e^{-|z|^2/4} \frac{1}{\sqrt{2\pi 2^{m'} m'!} \ell} z^{m'} e^{-|z|^2/4} \quad (42)$$

$$= \frac{1}{\sqrt{2\pi 2^m m!} \ell} \frac{1}{\sqrt{2\pi 2^{m'} m'!} \ell} \int_{\psi=0}^{2\pi} \int_0^\infty dr d\psi e^{-i(m-m')\psi} r^{m+m'+1} e^{-r^2} \quad (43)$$

$$= \frac{\delta_{m,m'}}{m!} m! = \delta_{m,m'} \quad (44)$$

Exercises. Prove the average area of each Landau orbital.

### Degeneracy

We have learnt that the energy of 2D charged particles is characterised by a quantum number  $n$ , which denotes the Landau level index. But the system hosts large degeneracy for each Landau level.

Semi-classically, the area of each electron orbital is  $2\pi\ell^2$ . This minimal surface plays the same role as the surface (action)  $h$  in phase space and therefore allows us to count the number of possible quantum states of a given (macroscopic) surface  $S$ :

$$N_\phi = \frac{S}{2\pi\ell^2}. \quad (45)$$

Notice that the flux density

$$n_B = \frac{1}{2\pi\ell^2} = \frac{B}{e/h} \quad (46)$$

which is nothing other than the magnetic field measured in units of the flux quantum  $h/e$ . Therefore, the number of quantum states in a Landau level equals the number of flux quanta.

In order to describe the LL filling it is therefore useful to introduce the dimensionless ratio between the number of electrons  $N_e$  and that of the flux quanta

$$\nu = \frac{N_e}{N_\phi} \quad (47)$$

The Landau level degeneracy can be made more precisely. Considering the Landau gauge, we have  $k_y = m2\pi/L_y, m \in [1, \dots, N_\phi]$  for the wave vector in the x-direction, one may count the number of states in a rectangular surface of length  $L_y$  and width  $L_x$ . In x-direction, max value of x is also bound by  $x_{max} = k_y \ell^2 = N_\phi 2\pi \ell^2 / L_y = L_x$ , so we have,

$$N_\phi = \frac{L_x L_y}{2\pi \ell^2}. \quad (48)$$

The LL degeneracy can also be obtained readily in the symmetric gauge. Consider a disk of radius  $R$  centered at the origin, and ask how many states lie inside it in a given Landau level. Taking, for simplicity, the lowest Landau level, the eigenstate  $|0, m\rangle$  has its weight located at the circle of radius  $r = \sqrt{2m\ell}$ . Thus the largest value of  $m_{max}$  for which the state falls inside the disk is given by  $m_{max} = R^2/2\ell^2 = \pi R^2/2\pi\ell^2$ , which is also the total number of eigenstates in the lowest Landau level that fall inside the disk. Thus, the degeneracy per unit area is  $(2\pi\ell^2)^{-1} = eB/h$ .

Remark. 1. Gauge. 2. Geometry.

## THEORY OF IQHE: LAUGHLIN'S PUMP PICTURE

In order to explain the quantization of the Hall resistivity, Laughlin considered a gedanken experiment as shown in Fig. 6. Here the two-dimensional surface is bent into a cylinder, and electrodes are placed at the edges of the cylinder. The radius of the cylinder is  $R$ , and the length is  $L$ . A magnetic field of constant magnitude is applied perpendicular to the cylindrical surface. The  $x, y$  coordinates are chosen as shown in the figure: the  $x$  axis is parallel to the axis of the cylinder, and the  $y$  axis is along the circumference. We place a solenoid on the axis of the cylinder, and a magnetic flux  $\Phi$  is generated inside this solenoid. The solenoid plays an important role in the following argument. (This placement of the solenoid, which is possible in a cold-atom system by “shaking the lattice”.) The magnetic field of the solenoid exists only inside the solenoid, and does not exist at the surface of the cylinder. However, the vector potential of this field has a finite value on the surface and affects the motion of electrons. This is the Aharonov-Bohm effect.

The vector potential of the magnetic field penetrating the cylinder surface is given by

$$\mathbf{A} = (0, Bx, 0) \quad (49)$$

and the vector potential generated by the solenoid is

$$\mathbf{A}_\Phi = (0, \Phi/(2\pi R), 0). \quad (50)$$

It can be deduced by

$$\Phi = \int dS \nabla \times \mathbf{A}_\Phi = \int d\mathbf{l} \cdot \mathbf{A}_\Phi = 2\pi R \times \Phi/(2\pi R) \quad (51)$$

(it should since there is no magnetic flux coming from on the cylinder).

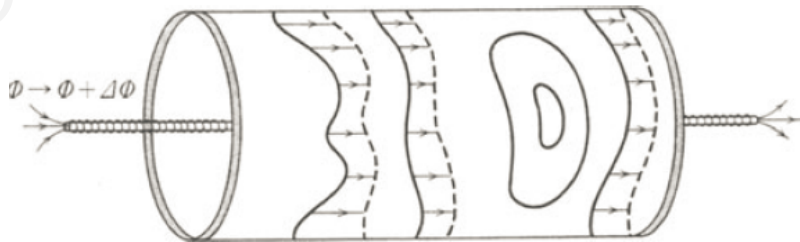


FIG. 6: Gedankenexperiment considered by Laughlin.

Let us see what happens when we change  $\Phi$ . By Faraday's law we know there will be an electric field  $\mathbf{E}$  induced along "e" such that

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt} \quad (52)$$

where  $C$  is a ring around the cylinder in  $y$  direction. As a result there will be a Hall current

$$\mathbf{j}_H = \sigma_H \hat{e}_z \times \mathbf{E} \quad (53)$$

Then the current across the fiducial line  $C$  is

$$\frac{dQ}{dt} = I = \int_C \mathbf{j}_H \cdot \hat{e}_x dl \quad (54)$$

$$= \int_C \sigma_H (\hat{e}_z \times \mathbf{E}) \cdot \hat{e}_x dl \quad (55)$$

$$= \sigma_H \int_C (\hat{e}_z \times \hat{e}_x) \cdot \mathbf{E} dl \quad (56)$$

$$= \sigma_H \int_C \hat{e}_y \cdot \mathbf{E} dl \quad (57)$$

$$= -\sigma_H \frac{d\Phi}{dt} \quad (58)$$

Here we get that, the transport rate of charge along the cylinder axis direction is related to the rate of flux, or equivalently the transport charge is related to the change of flux:  $\Delta Q = \sigma_H \Delta \Phi$ . Then we consider the flux changes "flux quanta",  $\Delta \Phi = h/e$ , then we have  $\Delta Q/e = \sigma_H (h/e^2)$ . Thus we reach the statement of the gedanken experiment: *as  $\Phi$  increases from  $\Phi$  to  $\Phi + 2\pi$  (a flux quanta), a charge  $\Delta Q = \sigma_H (h/e^2)$  will be transported (or "pumped") from the left edge to the right edge.*

The Hamiltonian is given by (the difference due to the gauge solenoid and the addition of a flux gauge potential.)

$$H = \frac{1}{2m} (\mathbf{p} - e\mathbf{A})^2 \quad (59)$$

$$= \frac{1}{2m} \left( p_x^2 + \left( p_y - eBx - e\frac{\Phi}{2\pi R} \right)^2 \right) \quad (60)$$

$$= \frac{1}{2m} \left( p_x^2 + \left( \frac{nh}{R} - eBx - e\frac{\Phi}{2\pi R} \right)^2 \right) \quad (61)$$

$$= \frac{1}{2m} \left( p_x^2 + (eB)^2 \left( x - \frac{1}{2\pi RB} [n2\pi h/e - \Phi] \right)^2 \right) \quad (62)$$

$$= \frac{1}{2m} \left( p_x^2 + (eB)^2 (x - x_n(\Phi))^2 \right) \quad (63)$$

Here we used the quantized condition  $p_y = \frac{2\pi\hbar}{R}n$ . We recover the harmonic oscillator problem by replacing the cyclotron center by  $x_n(\Phi)$ . So we get the following schematic picture of energy versus the  $x$  coordinate: Note that a change of  $\Delta\Phi = 2\pi\hbar/e$  for a particular  $n$  results in

$$x_n(\Phi+2\pi\hbar/e) = \frac{1}{2\pi RB}[n2\pi\hbar/e - (\Phi+2\pi\hbar/e)] = \frac{1}{2\pi RB}[(n-1)2\pi\hbar/e - \Phi] = x_{n-1}(\Phi) \quad (64)$$

so that to change the flux by  $2\pi$  (flux quanta) is equivalent to merely shifting or relabeling the whole picture  $n$  by  $n - 1$ . Thus only what happens at the edges  $x = 0$  and  $x = L$  will matter, using the fact that occupation is inherited. Thus we see that at the right edge some of the empty states that used to be above the Fermi line will now move below it: charge is “lost” on the right edge. On the left edge, some of the occupied states that used to be below the Fermi line will now move above it: charge is “gained” on the left edge. To compute the total transfer of charge from left to right, we merely have to count the number of occupied Landau levels:

$$\Delta Q = -e \times (\text{\#occupied Landau levels}) \quad (65)$$

which is a quantized number. Thus we have:

$$\sigma_H = \Delta Q \frac{e^2}{h} = (\text{\#occupied Landau levels}) \frac{e^2}{h} \quad (66)$$

Note that the number of occupied levels is a deterministic integer, not a random one.

## THEORY OF IQHE: LANDAUER PICTURE

Halperin was the first to try to explain the Hall effect by means of the imbalance of the edge currents at opposite edges of the sample. Later, this idea was developed by Buttiker. The Landauer-type and Laughlin's derivations of the Hall quantization are closely related. Both require the Fermi level to lie in the localized states in the interior. Both also require extended edge states. Laughlin's derivation implicitly assumes that the inner and outer edges are not coupled by disorder, which is equivalent to assuming a suppression of backscattering in the other formulation. The Landauer-type formulation is more general. (i) It more closely resembles experiments. (ii) Laughlin's derivation assumes that edge states are phase coherent over the entire length of the sample, which is not the case at nonzero temperatures.

### Edge states

The existence of edge states can be motivated classically. In a classical Hall bar with a uniform perpendicular magnetic field, charged particles move in circular motion, the cyclotron orbit. Suppose that our particles are negatively charged and therefore move in counter-clockwise direction. If one places a particle at the right edge of the sample, the particle will do a half-orbit until it collides with the edge. It can't leave the sample and it can also only do counter-clockwise rotational motion, so it will perform another half-orbit, but now it is further down on the edge, see Fig.7. The macroscopic effect is a local current

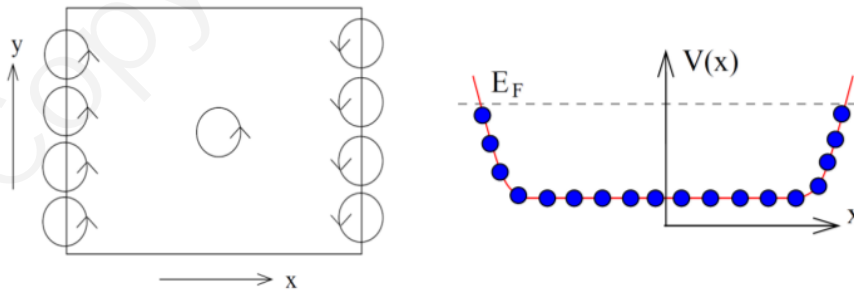


FIG. 7: Classical description of the edge states. Particles at the edge of the sample collide with the edge when moving in the cyclotron orbit and result in an edge current. Model of the sample's edges as a steeply rising potential, where the states below the Fermi level are occupied.

on either edge of the sample, with opposing directions, so the net current stays zero.

Edge states can also be treated quantum mechanically. To model the edge of the sample, one introduces a potential well that rises steeply at the edges. Since the states of the unperturbed system are localised in the x-direction, and by assuming that the potential varies slowly on the scale of the magnetic length, we can Taylor expand the potential around these center positions and drop all the constant terms. So the problem we will deal with is like

$$H = \frac{1}{2m}(p_x^2 + (eB)^2(x - k_y\ell^2)^2) + x \frac{\partial V}{\partial x} \quad (67)$$

This Hamiltonian can be solved exactly, by

$$H = \frac{1}{2m}(p_x^2 + (eB)^2(x - k_y\ell^2 + V'(x_n)\frac{1}{(eB)^2})^2) + k_y\ell^2 V'(x_n) - \frac{1}{(eB)^2} V'(x_n)^2 \quad (68)$$

The extra terms are all constant on the  $k_y$  momentum subspace so they transfer over to the energy eigenvalues, while the states will have the same form as the ones found earlier. The resulting energy eigenvalues are

$$E_n(k_y) = \hbar\omega_B(n + \frac{1}{2}) - \frac{1}{(eB)^2} V'(x_n)^2 + k_y\ell^2 V'(x_n) \quad (69)$$

They do depend on the y-momentum  $k_y$ , so we can calculate a finite group velocity of the wavepackets:

$$v_y = \frac{\partial E}{\hbar \partial k_y} = \frac{1}{eB} V'(x_n) \quad (70)$$

By looking again at the potential picture and using the fact that the states are still localised in x-direction, one sees that the states on the right edge propagate in negative y-direction and opposite for the left side. These edge states are chiral, they can only move in one direction. Since they are chiral, they are immune to back-scattering by impurities. In the classical picture, defects and impurities scatter incoming electrons into random directions, effectively decreasing the current.

#### Four-terminal

First we consider the ideal two-dimensional electron system shown in Fig. 8. The upper and lower edges are connected to electrodes, and the right and left sides of the sample are

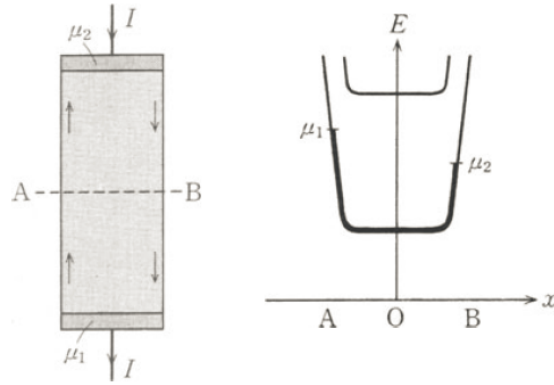


FIG. 8: Simple understanding based on Landauer picture.

defined by an infinitely high potential barrier. We consider a situation where the Fermi level is in between Landau levels, so the system is at the center of a quantum Hall plateau. The Fermi levels in the upper and lower electrodes may be different, for example the Fermi level in the lower electrode  $\mu_1$  may be higher than that in the upper electrode  $\mu_2$ . In this case the edge states at the left edge are occupied up to  $-\mu_1$ , but those at the right are occupied only up to  $\mu_2$ . This nonequilibrium distribution of electrons will not be equilibrated in an ideal system, owing to the conservation of momentum.

Now we observe that the group velocity of each Landau level is

$$v_n = \frac{dE_n}{\hbar dk} \quad (71)$$

The current from each level  $n$  is canceled out between left and right edges and the only part that contributes is the difference contributes to the net current. So the current is given by

$$I_n = e \int \frac{1}{2\pi} v_n dk \quad (72)$$

$$= e \int \frac{1}{2\pi} \frac{dE_n}{\hbar dk} dk \quad (73)$$

$$= e/h \int_{\mu_1}^{\mu_2} dE_n \quad (74)$$

$$= e/h(\mu_2 - \mu_1) = e^2/hV \quad (75)$$

so that the total current is given by

$$I = (\#\text{occupied Landau levels}) \frac{e^2}{h} V \quad (76)$$

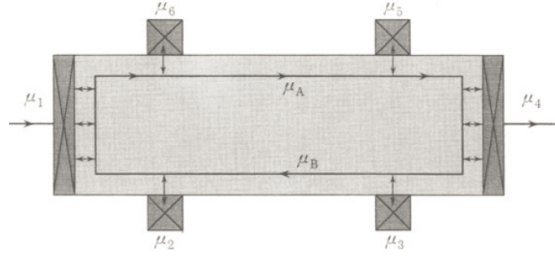


FIG. 9: Hall bar from the view point of edge states.

### Six-terminal

With the above preparation, next we discuss the general case similar to the experimental setup (Fig. 9). At the upper edge, the two electrodes have the same chemical potential  $\mu_5 = \mu_6 = \mu_A$ . Likewise, at the lower edge,  $\mu_2 = \mu_3 = \mu_B$ , where  $\mu_{A,B}$  are the chemical potentials of the upper and lower edges after equilibrium is attained at each edge.

Similarly, we assume that the left electrode has the chemical potential with the upper edge, and assume the electrons perfectly transport from electrode to the sample. So we have the current  $I = Ne/h(\mu_1 - \mu_4) = Ne/h(\mu_A - \mu_B)$ . The Hall voltage is  $V_H = (\mu_A - \mu_B)/e$ . So we get  $I/V_H = Ne^2/h$ , and  $N = (\# \text{occupied Landau levels})$ . Notice that it is self-consistent that, longitudinal resistance is zero:  $R_L = (\mu_2 - \mu_3)/eI = (\mu_5 - \mu_6)/eI = 0$ .

At last, we emphasize the astonishing feature of perfect transmission, which is independent of the length  $L$  (or more precisely of the aspect ratio  $L/W$ ) or the particular geometry of the sample, may be understood from the edge-state picture which we have introduced above. Perfect transmission is due to that the electron cannot be backscattered unless it is scattered to the opposite edge with inverse chirality. However, in a usual quantum Hall system, the opposite edges are separated by a macroscopic distance  $W$ , and backscattering processes are therefore strongly (exponentially) suppressed, which determines the spatial extension of quantum-mechanical state, and the macroscopic sample width  $W$ .

## THEORY OF IQHE: EXISTENCE OF PLATEAU, ROLE OF DISORDER

The theory of the IQHE, namely integral quantization of the Hall resistance, must answer the following questions: What is the origin of the quantization of the Hall resistance? What determines its value? Why is the quantization precise?

Let us take the classical formula for the Hall resistance

$$R_H = \frac{B}{\rho e}. \quad (77)$$

At the precise integer filling fraction, the longitudinal conductivity is zero, and the Hall conductivity is precisely the quantized value  $R_H = \frac{B}{ne} = \frac{h}{ne^2}$ , which is precisely the value of quantized Hall resistance. Is this the explanation of the Hall resistance? No, because it does not explain the plateaus. (In experiment, generically the chemical potential does sit in the degenerate band and the filling fraction is tuned continuously and is not quantized.)

While the system does have a gap under integer filling, we will need something that will preserve the special properties of the state even when we move away from the filling fraction which is precisely an integer. There happens to be a way out. An assumption made above, that of translational invariance, is not satisfied in the actual experimental system because of the unavoidable presence of disorder. Disorder, surprisingly, is crucial for the establishment of plateaus.

### The effect of disorder

*Tip: Anderson Localization. 1958 P. W. Anderson (for which he won the Nobel prize in 1977).*

Disorder obviously modifies the spectrum. Much work has been done on how the single electron eigenstates are affected by the presence of disorder. The resulting picture is shown schematically in Fig. 10. Disorder will spread out the energies in the band by having some regions where the potential is higher than average and some regions where the potential is lower than average. This spreads the sharp peak in the density of states into a broadened band. (These ideas were developed by Iordansky, Kazarinov and Luryi 1982, Prange and Joynt 1982, and Trugman 1983)

To develop a feel for the effect of disorder on single particle eigenstates, let us consider a disorder potential that is smooth on the scale of the magnetic length, which will allow us

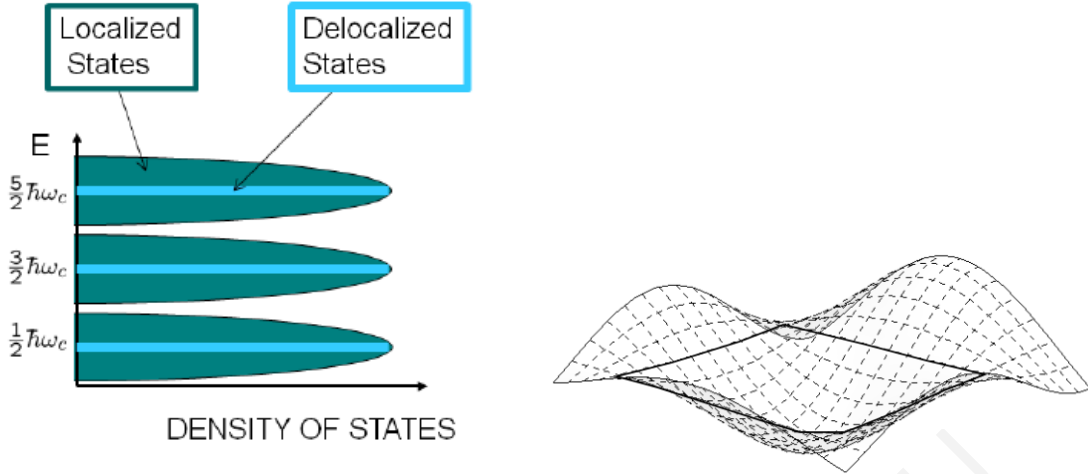


FIG. 10: The density of states for spin-polarized (or spinless) electrons in a magnetic field with disorder. The Landau bands are spread out, with localized eigenstates in the tails and extended eigenstates near the middle. Disorder potential configurations.

to use our semiclassical intuition. We imagine a potential landscape with hills and valleys. Landau levels are locally well defined for such a potential. For a given energy, an electron moves along an equipotential contour. Another way to see this is that the  $E \times B$  drift and  $E = -\partial V_{dis}(r)$ . Thus, the disorder potential traps the electron into a closed orbit. This, interestingly, is true independent of whether the potential is attractive (a potential well) or repulsive (a potential hill).

Since current tends to flow perpendicular to potential gradients (i.e., it is hall current), eigenstates tend to follow contours of constant potential. Thus many of the eigenstates at high and low energy will be trapped in local minima or maxima isolated in a hill or valley and circling the peak or bottom. (i.e. If the equipotential lines are closed, as it is the case for most of the equipotential lines in a potential landscape, an electron cannot move from one point to another one over a macroscopic distance. An electron moving on a closed equipotential line can therefore not contribute to the electronic transport, and the electron is thus localised. Notice that this type of localisation it different from other popular types.) The result is that the eigenstates at the edge of the band experience localization, whereas (at least one ) eigenstates near the center of the band are extended. When the chemical potential is anywhere in the localized states, then at low enough temperature, the electrons cannot move at all. Although there are states at this energy, they are all localized and

electrons cannot jump between them. Hence we expect in this case that the longitudinal conductance goes to zero (it is insulating).

The point that is not obvious, is why the Hall conductance should be precisely quantized. We will explain it in the following sections.

Exercises. Prove that, in the semiclassical picture, velocity satisfies “no source” condition, so that the disorder doesnot change the “conducting” electron.

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0 \quad (78)$$

$$(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \times \frac{\mathbf{B}}{B^2} = 0 \quad (79)$$

$$-\partial V \times \frac{\mathbf{B}}{B^2} - \mathbf{v} = 0 \quad (80)$$

$$\mathbf{v} = -\partial V \times \frac{\mathbf{B}}{B^2} \quad (81)$$

$$\partial \cdot \mathbf{v} = \partial \cdot \left( \frac{\mathbf{B}}{B^2} \times \partial V \right) \quad (82)$$

$$= \partial_i \left( \frac{\mathbf{B}}{B^2} \times \partial V \right)_i \quad (83)$$

$$= \frac{1}{B} \partial_i \epsilon_{i3j} \partial_j V = 0 \quad (84)$$

### Interpretation to experiment

We start with the situation of  $n$  completely filled LLs [column (a) of Fig. 11], which we have extensively discussed above: the Landau level  $n$  (and its potential landscape) is unoccupied. In a six-terminal measurement, one therefore measures the Hall resistance  $R_H = h/e^2 n$  and a zero longitudinal resistance. Therefore, the quantum Hall system at integer filling factors is therefore a very unusual electron liquid: it is indeed a bulk insulator with perfectly conducting (non-dissipative) edges.

In column (b) of Fig. 11, we represent the situation where the Landau Level  $n$  gets moderately filled by electrons when the magnetic field  $B$  is decreased. These electrons in  $n$  populate preferentially the valleys of the potential landscape, or more precisely the closed equipotential lines that enclose these valleys. The electrons in the Landau Level  $n$  are thus (classically) localised somewhere in the bulk and do not affect the global transport characteristics, measured by the resistances, because they are not probed by the sample contacts. Therefore, the Hall resistance remains unaltered and the longitudinal resistance

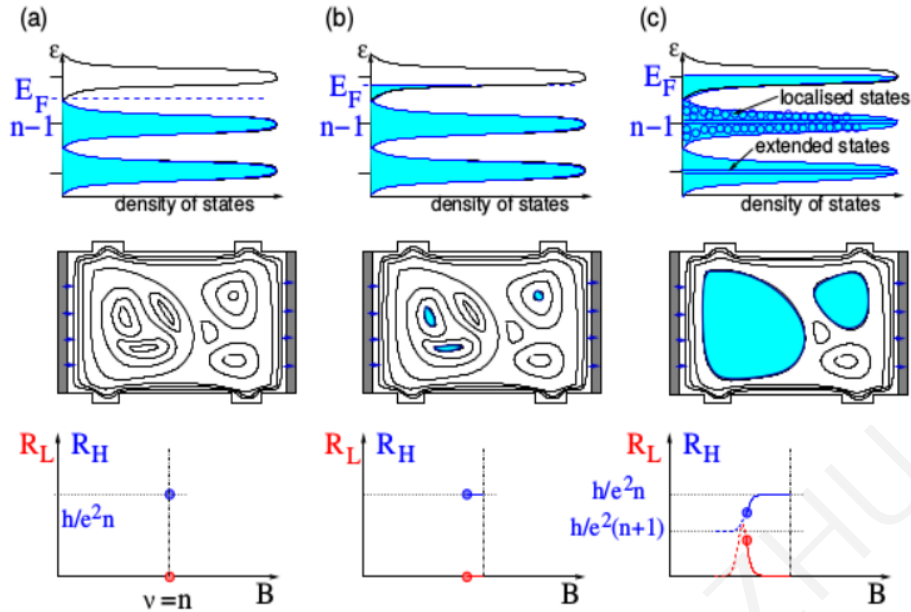


FIG. 11: The disorder-broadened density of states is shown in the first line for increasing fillings (a) - (c) described by the Fermi energy. The second line represents the disorder-potential landscape the valleys of which become successively filled with electrons when increasing the filling factor, i.e. when lowering the magnetic field at fixed particle number. The third line shows the corresponding Hall (blue) and the longitudinal (red) resistance measured in a six-terminal geometry, as a function of the magnetic field. The first figure in column (c) indicates that the bulk extended states are in the centre of the DOS peaks, whereas the localised states are in the tails.

remains zero despite the change of the magnetic field. This is the origin of the plateau in the Hall resistance.

If one continues to lower the magnetic field, the regions of the potential landscape in the Landau Level  $n$  occupied by electrons become larger, and they are eventually enclosed by equipotential lines that pass through the bulk and that connect the opposite edges. In this case, an electron injected at the left contact and travelling a certain distance at the upper edge may jump into the state associated with this equipotential line and thus reach the lower edge. Due to its chirality, the electron is then backscattered to the left contact, which causes an increase in the longitudinal resistance. Indeed, if one measures the resistance between the two contacts at the lower edge, a potential drop is caused by the electron that leaks in from this equipotential connecting the upper and the lower edge. It is this potential drop

that causes a non-zero longitudinal resistance. At the same moment the Hall resistance is no longer quantised and jumps to the next (lower) plateau, a situation that is called plateau transition. This situation of electron-filled equipotential lines connecting opposite edges, which are thus extended states [see first line of Fig. 11(c)] as opposed to the bulk localized states, arises when the Landau Level  $n$  is approximately half-filled.

### Plateau transitions and scaling laws

The physical picture presented above suggests that the plateau transition in the Hall resistance is related to a percolation transition, where initially separated electron-filled valleys start to percolate between the opposite sample edges beyond a certain threshold of the filling. Because of the second-order character of a percolation transition, this scenario suggests that the plateau transition is a second-order quantum phase transition described by universal scaling laws, where the control parameter is just the magnetic field  $B$ .

The phase transition occurs at the critical magnetic field  $B_c$  and is characterised by an algebraically diverging correlation length

$$\xi \sim \frac{1}{|B - B_c|^\nu} \quad (85)$$

where  $\nu$  is called the critical exponent.

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