

Notes on the Ising model

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Contents

Mean-field solution	2
Weiss Mean-Field Theory	2
Critical exponents in phase transitions	4
critical behavior 1	4
critical behavior 2	5
critical behavior 3	5
critical behavior 4	6
Landau phase transition theory	10
Order parameters and symmetry spontaneously broken	12
More on discrete symmetry	13
More on continuous symmetry	14
Fluctuating field and ϕ^4 theory	15

MEAN-FIELD SOLUTION

Weiss Mean-Field Theory

The simplest approximate treatment of spin systems in general is the mean-field theory (MFT) approximation. The general idea is to replace the many-spin system by an effective one-spin Hamiltonian in the presence of an effective external field produced collectively by the remaining spins. The approximation is valid away from the transition point, deep in the classically well-ordered and classically disordered states, where fluctuations are small.

To implement Weiss mean-field approximation on the classical spin model, we assume long-range magnetic order, characterized by a magnetization proportional to $\langle \sigma_i \rangle$, with spin then given by

$$\sigma_i = \langle \sigma_i \rangle + (\sigma_i - \langle \sigma_i \rangle) \quad (1)$$

a sum of the mean-field value and (presumed) small classical fluctuations. Inserting this into the spin Hamiltonian in the presence of an external field and neglecting the small fluctuations terms beyond first order, we obtain

$$\begin{aligned} H_{MFT} &= -J \sum_{\langle ij \rangle} [\langle \sigma_i \rangle + (\sigma_i - \langle \sigma_i \rangle)] [\langle \sigma_j \rangle + (\sigma_j - \langle \sigma_j \rangle)] + B \sum_i \langle \sigma_i \rangle + (\sigma_i - \langle \sigma_i \rangle) \\ &\approx J \sum_{\langle ij \rangle} [-\langle \sigma_i \rangle \langle \sigma_j \rangle + \langle \sigma_i \rangle \sigma_j + \langle \sigma_j \rangle \sigma_i] + B \sum_i \sigma_i \\ &= \sum_i B_{eff} \cdot \sigma_i - J \sum_{\langle ij \rangle} \langle \sigma_i \rangle \langle \sigma_j \rangle \end{aligned} \quad (2)$$

where the effective Weiss field is

$$B_{eff} = B + 2J \sum_j \langle \sigma_j \rangle = B + Jz \langle \sigma \rangle \quad (3)$$

where z is the number of nearest neighbors, e.g. $z = 2$ for 1d, $z = 4$ for 2d, $z = 8$ for 3d.

It quite clearly gives a self-consistent mechanism to induce magnetic order, $\langle \sigma_i \rangle \neq 0$, even for a vanishing external magnetic field. The Weiss field on spin σ_i is generated by the neighboring spins. Since the above mean-field Hamiltonian is for a single spin, it can be solved exactly utilizing the analysis below, by including an implicit self-consistency condition through B_{eff} .

Focussing on the ferromagnetic state ($J > 0$), we take $\langle \sigma_i \rangle = m$ to be spatially uniform. Let us recall magnetization density along the applied field

$$Z_1(\beta) = e^{-\beta H} = e^{\beta \frac{Jzm^2}{2}} \sum_{s=\pm 1} e^{-\beta s B_{eff}} = 2e^{\beta \frac{Jzm^2}{2}} \cosh(\beta B_{eff}) \quad (4)$$

For N noninteracting spins $Z_N = (Z_1)^N$, the free energy $F_N = NF_1$, and

$$m(B) = -\frac{\partial F_N}{\partial B} = \tanh \beta B_{eff} \quad (5)$$

Alternatively, the magnetization can also be obtained by

$$m(B) = \langle \sigma_i \rangle = Tr[\sigma_i e^{-\beta H}] = \frac{e^{\beta B_{eff}} - e^{-\beta B_{eff}}}{e^{\beta B_{eff}} + e^{-\beta B_{eff}}} = \tanh \beta B_{eff} \quad (6)$$

This is the self-consistent equation that should be satisfied by $m(B)$. One can solve it, for example by plotting (see Fig. 1). The non-trivial solution depends on the slope of T , which determines the critical temperature:

$$T_c = \frac{zJ}{k_B}. \quad (7)$$

It is clear that for sufficiently high $T > T_c$, and zero external field $B = 0$, there is only a single trivial paramagnetic solution $m = 0$. However, for $T < T_c$, and there is also a nontrivial, ferromagnetic $m > 0$ solution, that can be shown to minimize the free energy for $T < T_c$. The slope of the $m \neq 0$ solution will determine the real ground state.

Here, for $T > T_c$, the system is paramagnetic, and the ground state has the symmetry with the Hamiltonian. For example, one can check the Z_2 symmetry by sending $\sigma_i \rightarrow -\sigma_i$, the Hamiltonian is unchanged. The paramagnetic state preserves this symmetry. In contrast, for $T < T_c$, $m_0 \neq 0$ and the system is ferromagnetic. Z_2 symmetry is not conserved any more, which means spin-up is not equivalent to spin-down. So the symmetry of the ground state is lower than the microscopic Hamiltonian. This phenomenon is dubbed as spontaneously symmetry breaking. We will return here in the discussion of Landau phase transition theory.

It can be verified that above expressions display the correct quantum and classical limits. In the latter, classical limit $B \ll k_B T$ the result reduces to Curie linear susceptibility

$$\chi_C(B=0, T) = \left. \frac{\partial m}{\partial B} \right|_{B \rightarrow 0} \rightarrow \frac{C}{T} \quad (8)$$

with C the Curie constant and $m \sim \chi_C(T)B$ exhibiting a linear response in this regime. This $1/T$ linear susceptibility behavior is a generic experimental signature of independent

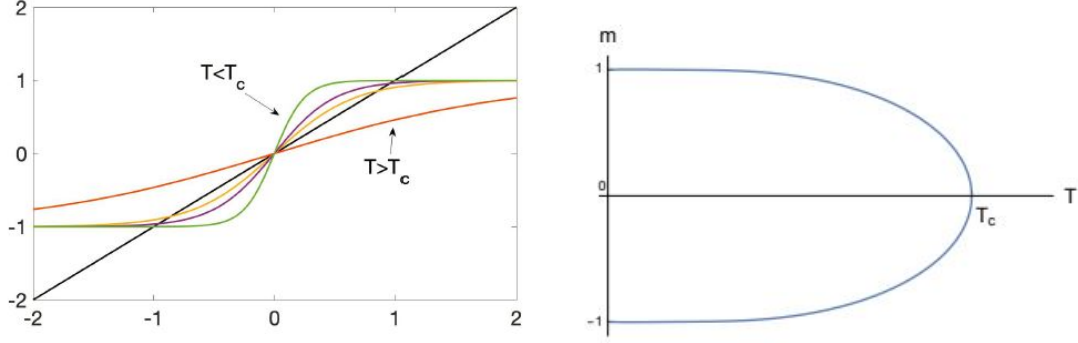


FIG. 1: Solution of mean-field self-consistent equation: $m(B) = \tanh \beta(B + Jzm)$ and the mean-field phase diagram of Ising model.

local moments, with the amplitude C a measure of the size of the magnetic moment and the associated spin. At finite T the susceptibility is finite and paramagnetic (i.e., magnetization is along the applied magnetic field and vanishes with a vanishing field), only diverging at a vanishing temperature. This captures the fact that in a classical regime, as $T \rightarrow 0$ a nonzero magnetization is induced in response to an infinitesimal field, as disordering thermal fluctuations vanish.

Critical exponents in phase transitions

Next we dive into the discussion of critical behavior around the phase transition point, and analyze the criticality.

critical behavior 1

For $T < T_c$ close to T_c , we assume magnetization is a small value $m_0 \rightarrow 0$. In this limit, we can expand $\tanh x = x - x^3/3 + \dots$, and get

$$m_0 \approx m_0 Jz\beta - \frac{1}{3}(\beta Jz m_0)^3 + \dots \quad (9)$$

Dividing by m_0 , we can solve as

$$m_0 = \pm \sqrt{3} \left(\frac{k_B T}{zJ} \right)^{3/2} \left(\frac{zJ}{k_B T} - 1 \right)^{1/2} \quad (10)$$

Using $T_c = zJ/k_B$, we have

$$m_0 = \pm\sqrt{3}\left(\frac{T}{T_c}\right)^{3/2}\left(\frac{T_c}{T} - 1\right)^{1/2} \quad (11)$$

Here, Mean-field theory (correctly) predicts that the magnetization vanishes as a power law with temperature. The critical exponent related to the magnetization is conventionally called $\beta = 1/2$.

In comparison, we can compare it with the Ising magnet in 3d, the magnetization vanishes as $M \sim (T_c - T)^{0.315}$, $T < T_c$; and in the two-dimensional Ising model, the exact computations give $M \sim (T_c - T)^{1/8}$, $T < T_c$.

critical behavior 2

In the ferro-magnetic phase, we consider thermal susceptibility per spin

$$\chi(T, B) = \left. \frac{\partial m}{\partial B} \right|_T \quad (12)$$

We expand the self-consistent equation like

$$m = \tanh(\beta B + zJ\beta m) \approx \beta B + \frac{\beta}{\beta_c} m - \frac{1}{3}(\beta B + zJ\beta m)^3 \quad (13)$$

$$\Rightarrow \beta B \approx \left(1 - \frac{T_c}{T}\right)m + \frac{1}{3}\left(\frac{T_c}{T}\right)^3 m^3 \quad (14)$$

We can calculate $\partial B/\partial m$ first and then calculation the susceptibility in the case of $B = 0$:

$$\chi(T, B = 0) = \left[\frac{\partial B}{\partial m}\right]^{-1} = \left[kT\left(\frac{T_c}{T}\right)^3 m_0^2 + kT\left(1 - \frac{T_c}{T}\right)\right]^{-1} \quad (15)$$

The susceptibility diverges with a power law both above and below the critical temperature T_c :

$$\chi(T) = \frac{1}{A_{\pm}|T - T_c|^{\gamma}} \quad (16)$$

and the critical exponent is $\gamma = 1$.

critical behavior 3

Again, from $m = \tanh\beta(B + Jzm)$ and $k_B T_c = Jz$ near the critical T_c , we have the magnetization in the non-zero magnetic field

$$m = \tanh\left(m + \frac{B}{zJ}\right) \approx m + \frac{B}{zJ} - \frac{m^3}{3} + \dots \quad (17)$$

Hence, $m \sim B^{1/3}$. This gives a critical exponent $\delta = 3$.

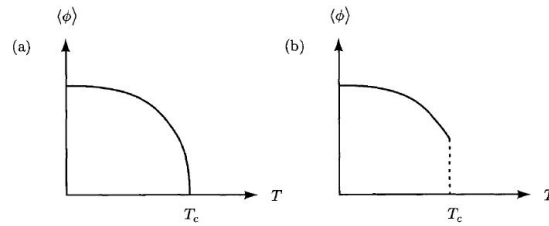


FIG. 2: Order parameter changes around the critical point.

critical behavior 4

Homework: Please calculate the specific heat satisfy with

$$C = \begin{cases} 0, T \rightarrow T_c^+ \\ 3NkT, T \rightarrow T_c^- \end{cases} \quad (18)$$

So there is no critical behavior in specific heat, or exponent $\alpha = 0$.

At last, several remarks are given in order. Mean-field theory correctly describes the qualitative features of most phase transitions and, in some cases. Since mean-field theory replaces the actual configurations of the local variables (e.g. spins) by their average value, it neglects the effects of fluctuations about this mean. These fluctuations may or may not be important. The more spins that interact with a particular test spin, the more the test spin sees an effective average or mean field. If the test spin interacts with two neighbors, the averaging is minimal and the fluctuations are large and important. The number of spins producing the effective field increases with the range of the interaction and with the dimension. Thus we deduce that mean-field theory is a good approximation in high dimensions but fails to provide a quantitatively correct description of second-order critical points in low dimensions.

Before proceeding, let us review some simple facts about phase transitions. At high temperatures, there is no order, and the order parameter m is zero. At a critical temperature, T_c , order sets in so that, for temperatures below T_c , m is nonzero. If m rises continuously from zero, as shown in Fig. 2, the transition is second order. If order parameter jumps discontinuously to a nonzero value just below T_c , the transition is first order.

We have seen that the thermal quantities becomes singular near the critical point (temperature). And we have defined four exponents $(\alpha, \beta, \gamma, \delta)$ related to physical quantities.

In the traditional way, there are two more critical exponents that are usually discussed. One is called ν , it appears in the correlation length in the correlation function $G(r) \sim \frac{1}{r} e^{-r/\xi}$:

$$\xi \sim |T - T_c|^{-\nu} \quad (19)$$

The mean-field value is $\nu = 1/2$. The other one is η , and it also appears in the Green's function at the critical point $G(r) \sim r^{-d+2-\eta}$. Its mean-field value is $\eta = 0$.

Traditionally, people conclude that the above six critical exponents satisfy the relationships

$$\alpha + 2\beta + \gamma = 2, \textit{Rushbrooke} \quad (20)$$

$$\gamma = \beta(\delta - 1), \textit{Widom} \quad (21)$$

$$\gamma = \nu(2 - \eta), \textit{Fisher} \quad (22)$$

$$\nu d(2 - \alpha), \textit{Josephson} \quad (23)$$

So only two independent critical exponents.

Nowadays, it has been established that the critical exponents should depend on the spatial dimension, the symmetry of the order parameter, and the symmetry and range of interactions, but not on the detailed form and magnitude of interactions. Thus, there are universality classes, and all transitions in the same universality class have the same critical exponents. For example, all transitions in which the order parameter has up-down symmetry ($n = 1$, Ising) should have the same exponents. To study these universality is the key problem in the study of phase transition.

The comparison between mean-field theory and experiments, please see Fig. 3. Critical exponents for most experimental second-order transitions differ from those predicted by mean-field theory.

Furthermore, experiments reveal that, the above critical exponents look like universal, independent of experimental details. For some quite different systems, they may share the very similar critical behavior! (Be caution: the experimental data is actually not so good.) This drives the ‘‘universality hypothesis’’: Two different systems with the same dimension d , the same symmetry may take the same critical behavior, independent of the form of interactions or other details.

Here we further show one example: The Ising universality class is characterized by the same critical exponents in the van der Waals fluids near the critical gas-liquid phase tran-

Exponent	α	β	γ	ν	η
Property	specific heat	order parameter	susceptibility	coherence length	correlation function
Definition	$C \sim t^{-\alpha}$	$\langle \phi \rangle \sim t^\beta$	$\chi \sim t^{-\gamma}$	$\xi \sim t^\nu$	$G(q) \sim q^{-2+\eta}$
Mean-field	0	0.5	1	0.5	0
3D theory					
$n = 0$ (SAW)	0.24	0.30	1.16	0.59	
$n = 1$ (Ising)	0.11	0.32	1.24	0.63	0.04
$n = 2$ (xy)	-0.01	0.35	1.32	0.67	0.04
$n = 3$ (Heisenberg)	-0.12	0.36	1.39	0.71	0.04
Experiment					
3D $n = 1$	$0.11^{+0.01}_{-0.03}$	$0.32^{+0.16}_{-0.04}$	$1.24^{+0.16}_{-0.04}$	$0.63^{+0.04}_{-0.04}$	$0.03 - 0.06$
3D $n = 3$	$0.1^{+0.03}_{-0.04}$	$0.34^{+0.04}_{-0.04}$	$1.4^{+0.07}_{-0.07}$	$0.7^{+0.03}_{-0.03}$	
2D $n = 1$	$0.0^{+0.01}_{-0.03}$	$0.3^{+0.04}_{-0.04}$	$1.82^{+0.07}_{-0.07}$	$1.02^{+0.07}_{-0.07}$	
Experiments on 3D $n = 1$ compiled from liquid-gas, binary fluid, ferromagnetic, and antiferromagnetic transitions.					
Experiments on 3D $n = 3$ transitions compiled from some ferromagnetic and antiferromagnetic transitions.					
Experiments on 2D $n = 1$ compiled from some antiferromagnetic transitions.					

FIG. 3: Some critical exponents from theory and experiments (Copyright from *Principles of condensed matter physics*, Cambridge University Press).

sition. It is summarized below. The liquid-gas transition has much in common with the

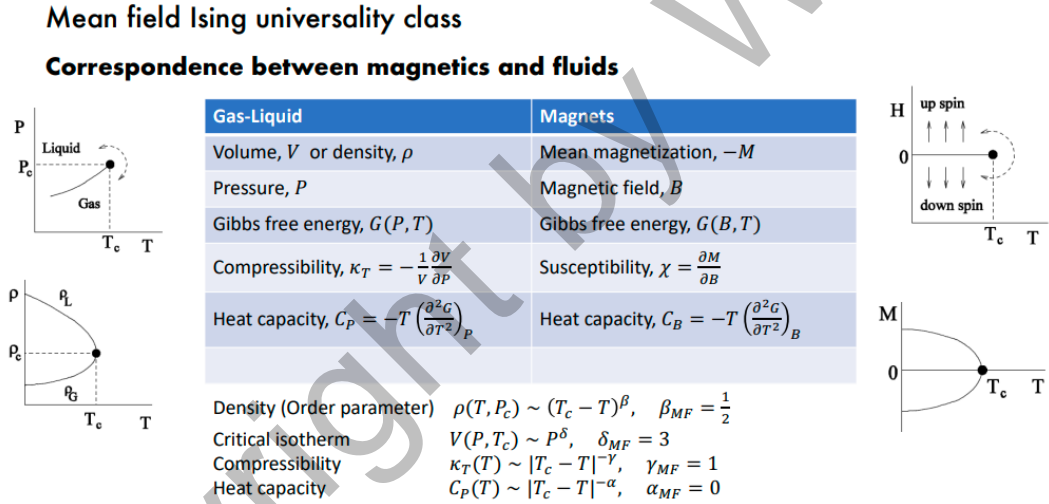


FIG. 4: Liquid-gas transition v.s. PM-FM transition.

magnetic transition in an Ising model. We compare the phase diagram in the $h - T$ plane for the Ising model to the phase diagrams in the $p - T$ planes for a fluid near its critical point. In both the Ising model and the fluid, there is a coexistence curve, terminating at a critical point, along which two distinct but equal free energy phases coexist, and in both it is possible to go continuously around the critical point from one coexisting phase to the other by appropriately varying h and T . The coexistence curve for the magnet is a straight line $h = 0, T < T_c$, whereas that for the fluid is in general curved. The inversion ($m \rightarrow -m$) symmetry of a magnet forces the coexistence line to be the line $h = 0$ and the critical point

value of h and m both to be zero. There is no such symmetry in a fluid and no special values of the critical point parameters p_c, T_c .

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LANDAU PHASE TRANSITION THEORY

Here we begin to discuss the Landau's continuous phase transition theory. There are two very important concepts: Order parameter and symmetry spontaneously broken, which are the cornerstone of condensed matter physics.

Let us recall the implicit self-consistent MFT Eq. 5 for $m(B, T)$ is actually a saddle-point equation for the free-energy density $f(m, B, T)$ with respect to m , i.e., corresponds to $\partial f / \partial m = 0$. As we are interested in the behavior near the critical point, where the magnetization is small, $m \ll 1$. Thus, Eq. 5 lead to a free energy that is quartic polynomial in the magnetization m , with a quadratic coefficient and quartic one a positive constant.

While above mean-field analysis relies on a specific microscopic model, as was first argued by Lev D. Landau (1937), above mean-field predictions are much more universal and are a consequence of continuous phase transition. Guided by general symmetry principles, Landau postulated a phenomenological theory, i.e. near a continuous phase transition the free energy density exhibits a generic analytic expansion in powers of an order parameter, the magnetization, in the case of a PM-FM transition (see free energy of Eq. 4),

$$\begin{aligned}
 f &= -k_B T \ln Z_1 = \frac{Jzm^2}{2} - k_B T \ln 2 - k_B T \ln[\cosh(\beta B_{eff})] \\
 &= \frac{Jzm^2}{2} - k_B T \ln 2 - k_B T \left[\frac{(\beta B_{eff})^2}{2} - \frac{(\beta B_{eff})^4}{12} + \dots \right] \\
 &\stackrel{B_{eff}=mzJ}{=} \frac{Jzm^2}{2} - k_B T \ln 2 - k_B T \left[\frac{(\beta mzJ)^2}{2} - \frac{(\beta mzJ)^4}{12} + \dots \right] \\
 \Rightarrow f &= f_0 + a(T)\mathbf{m}^2 + b(T)\mathbf{m}^4 \tag{24}
 \end{aligned}$$

where $a(T) = \frac{k_B T_c}{2T}(T - T_c) = a_0(T - T_c)$ and $b = \frac{k_B T_c^4}{12T^3}$.

The form is dictated by the time-reversal symmetry of the Hamiltonian for $B = 0$ (for Ising case, $m \rightarrow -m$ is a symmetry for $B = 0$, dictating that no odd powers of m appear in f), with coefficients smooth functions of T , and, crucially $a(T) = a_0(T/T_c - 1)$, changing sign to $a(T < T_c) < 0$. In the presence of magnetic field, one can add a term hm to the free energy, which break the even parity of m .

In the Ising case for $a(T > T_c) > 0$, $f(m)$ is well-approximated by a parabola, with a single minimum at the origin, $m = 0$ (see Fig. 5). In contrast, for $a(T < T_c) < 0$, the free energy develops a symmetric double-well form, minimized by a finite magnetization, $m_0 = a/b \sim |T - T_c|^{1/2}$. Thus, this generic Landau theory indeed predicts the phenomenology

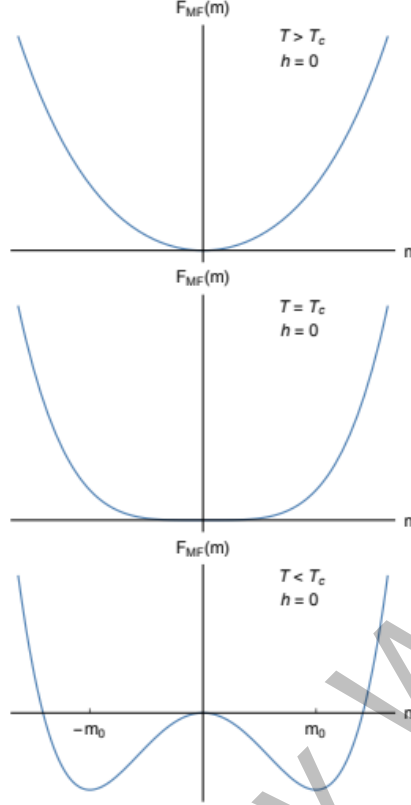


FIG. 5: Mean field solution of the free energy as a function of magnetization at zero field at various temperatures. As we cool the system below T_c , the free energy changes smoothly to a symmetry breaking case, thereby making the $m = 0$ solution unstable and two new stable solutions at $m = \pm m_0$ appear.

near T_c found above.

In the discussion of susceptibility, we can change the form of free energy to

$$f = f_0 + a(T)m^2 + b(T)m^4 - hm + \dots \quad (25)$$

The configurations of minimum F are spatially uniform and so require that

$$\frac{\partial f}{\partial m} = 0 \quad (26)$$

yields $2am + 4bm^3 = h$. This again relates the couplings in the Landau theory. When $h = 0$, we have $m = \pm\sqrt{\frac{-a}{2b}}$. If we assume b is constant, we have $a = \frac{T-T_c}{T_c}$ when $T < T_c$ to satisfy Eq. 11. The susceptibility is obtained by $\chi(B, T) = \frac{\partial m}{\partial B} = \frac{1}{2a+8bm^2}$, so the $\chi(B = 0, T) \sim 1/|T - T_c|$ can be obtained. Also, at $T = T_c$, we have $h = 4bm^3$, which is consistent with $m \sim h^{1/3}$. So the Landau mean-field theory is totally consistent. *Landau*

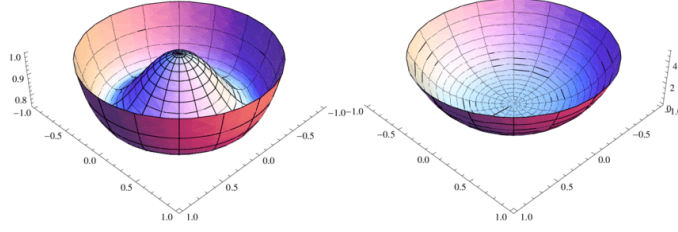


FIG. 6: Left is potential of free energy in symmetry breaking phase and the right is that in symmetric phase. In the symmetry breaking phase, the low-energy excitation is gapless Goldstone mode.

theory is remarkable in that, under the simple assumptions that the order parameter is small and uniform near T_c , it yields a wealth of information about phase transitions.

A new crucial ingredient will arise for the case of a multi-component vector order-parameter, \vec{m} . While MFT exponents remain the same, the Landau free-energy potential exhibits zero-energy (the so-called) Goldstone modes, corresponding to reorientation of the order parameter, that is, the motion along the minimum of the “Mexican hat” potential. This case is different from the discrete Z_2 order in Ising.

Order parameters and symmetry spontaneously broken

With all above preparation, here we give some formal definitions:

Order parameter.— It is a parameter to indicate a phase transition. It usually relates to some physical observable. Example is magnetization m in the Ising transition.

Symmetry spontaneously breaking.— Landau’s theory on phase transition is based on transition from HIGH symmetry state to a LOW symmetry state. This symmetry reducing process is called symmetry spontaneously breaking. In the above PM-FM transition, PM phase preserves the Z_2 transition while FM phase breaks this symmetry. So the Ising transition is a Z_2 transition.

Goldstone mode.— There are two symmetry breaking transition. One is discrete symmetry breaking, as PM-FM transition. The other type is continuous symmetry breaking. The free energy changes is like Fig. 6. In the symmetric breaking phase, the low-energy excitation is gapless, which is called Goldstone mode. Examples include phonon, magnon, etc.

Here we would like to give some more examples of phase transitions.

- Heisenberg Ferromagnet.

Order Parameter in the Heisenberg ferromagnet (in, say, 3D), the spin can point anywhere in space. Hence the magnetization is a vector, defined in the same way as for the Ising Model:

$$\mathbf{m}_i = \langle \mathbf{S}_i \rangle \quad (27)$$

The free energy becomes:

$$f = f_0 + a(T)\mathbf{m}^2 + b(T)\mathbf{m}^4 - h \cdot \mathbf{m} + \dots \quad (28)$$

Any free energy must be invariant under a rotation

$$\mathbf{m}_i \rightarrow \mathbf{R} \cdot \mathbf{m}_i \quad (29)$$

In this case, rotation is a continuous symmetry (i.e. SO(3) in 3D). This is the primary difference between Ising and Heisenberg: the Ising magnet breaks up-down symmetry, while the Heisenberg magnet breaks rotational symmetry: M can point anywhere in space.

- Ferroelectricity.

Order Parameter in polarization or electric dipole moment, which is from the inversion symmetry spontaneously breaking. Around the phase transition temperature, the crystal structure should change (the location of ions changes).

Material BaTiO₃ is shown in Fig. 7. Above T_c it is P_m3_m while below T_c it is P_4mm . Around the phase transition, Ti and O move along $+Z -Z$ direction.

More on discrete symmetry

The simplest discrete group is the group Z_2 consisting only of two elements: the identity and an element whose square is the identity $\{1, -1\}$. Realizations of this group include the group of reflections about a plane, time reversal, and so on. Z_2 symmetry is broken in any

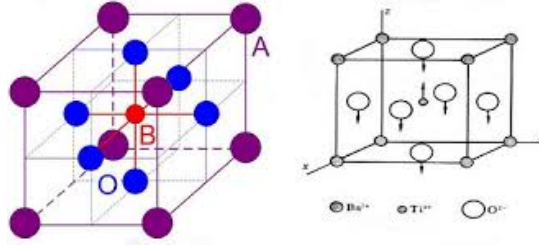


FIG. 7: BTO crystal structure changes at T_c , resulting in a spontaneous polarization.

phase transition in which there are two and only two equivalent ordered states characterized by order parameters that can be chosen to differ only in sign. Though Z_2 symmetry is the most important and most common discrete symmetry encountered in condensed matter physics, there are, of course, many others. A particularly useful hierarchy of discrete groups for studying properties of discrete symmetry is the set of groups Z_N of integers under addition modulo N .

A class of models with Z_N symmetry, called clock models, can be defined by associating with each site on a lattice a spin variable $s_i = (\cos(2\pi n_i/N), \sin(2\pi n_i/N))$ of unit length:

$$H = -J \sum_{n_i, n_j} \cos(2\pi(n_i - n_j)/N) \quad (30)$$

where $n_i = 0, \dots, N - 1$.

More on continuous symmetry

The simplest continuous group is the two-dimensional orthogonal group O_2 of rotations in a two-dimensional plane. Since a two-dimensional vector is equivalent to a complex number, the group O_2 is isomorphic to the group $U(1)$ of transformations of the phase of a complex number. The symmetry associated with these groups is often called xy-symmetry because rotations are usually done in the xy-plane. The group O_2 is of enormous pedagogical importance in the discussion of KT-transition. The simplest realization of a system with O_2 symmetry is an easy-plane ferromagnet in which spins are confined by crystal fields to lie in the xy-plane. The $O(2)$ or xy-model can be reexpressed in terms of a local angle variable by setting $s_i = (\cos \theta_i, \sin \theta_i)$:

$$H = -J \int d\theta_i \theta_j \cos(\theta_i - \theta_j) \quad (31)$$

where θ_i is a continuous variable. Please be aware the different in comparison with Z_N clock model.

The most important phase with broken $U(1)$ symmetry is superfluid helium. The superfluid phase is characterized by a macroscopic condensate wave function with an amplitude and a phase.

The group O_3 of rotations in three dimensions is another continuous group of considerable importance. Physical realizations of systems with this symmetry include Heisenberg ferromagnets and antiferromagnets on lattices where crystal fields aligning spins along crystal axes are unimportant.

FLUCTUATING FIELD AND ϕ^4 THEORY

The long-range character of critical fluctuations and the universality of the critical properties suggest that it is possible to calculate the critical behavior by a phenomenological field theory rather than a microscopic model. One may neglect the lattice completely and describe the partition function of the system near T_c by a functional integral over a continuous local order field $\phi(x)$. The energy functional is assumed to have a Taylor expansion in the order field $\phi(x)$ and in its gradients. Here we derive this phenomenological theory, which will appear in our future study.

Let us start from the partition function

$$\begin{aligned}
Z &= \sum_{\sigma_1=\pm, \dots, \sigma_N=\pm} \exp[-\beta \sum_{\langle ij \rangle} \sigma_i \sigma_j] = \sum_{\{\sigma_i\}} \exp[-\sum_{\langle ij \rangle} \sigma_i K_{ij} \sigma_j] \\
&= \text{const.} \int d\mathbf{v} \sum_{\{\sigma_i\}} e^{-\frac{1}{2} \sum_{ij} v_i (K_{ij}^{-1}) v_j + \sum_i \sigma_i v_i} \\
&= \text{const.} \int Dv e^{-\frac{1}{2} \sum_{ij} v_i (K_{ij}^{-1}) v_j} \prod_i (2 \cosh(v_i)) \\
&\stackrel{\phi_i = \sum_j K_{ij}^{-1} v_j}{=} \text{const.} \int D\phi e^{-\sum_{ij} \phi_i K_{ij} \phi_j + \sum_i \ln \cosh(2 \sum_j K_{ij} \phi_j)}
\end{aligned} \tag{32}$$

where we introduce a auxiliary field v_i and the Gaussian integral as

$$\int d\mathbf{v} e^{-\frac{1}{2} \vec{v} \cdot A \cdot \vec{v} + \vec{v} \cdot \vec{j}} = (2\pi)^{N/2} [\det A]^{-1/2} e^{\frac{1}{2} \vec{v} \cdot \vec{j}} \tag{33}$$

Then we assume the field is small $|\phi(x)| \ll 1$, and the spatial profile of the field is smooth, we can expand $\ln \cosh(x) = x^2/2 - x^4/12 + \dots$. If writing in the Fourier space,

$\phi_i = \frac{1}{N} \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}_i} \phi(\mathbf{k})$ and $K_{ij} = \frac{1}{N} \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot(\mathbf{r}_i-\mathbf{r}_j)} K(\mathbf{k})$, we obtain that

$$S[\phi] = \sum_{\mathbf{k}} \phi_{-\mathbf{k}} [c_0 + c_2 k^2] \phi_{\mathbf{k}} + c_4 \sum_{k_1, k_2, k_3, k_4} \delta_{\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4=0} \phi_{\mathbf{k}_1} \phi_{\mathbf{k}_2} \phi_{\mathbf{k}_3} \phi_{\mathbf{k}_4} + \dots \quad (34)$$

where we used $K(k) = K(0) + \frac{1}{2}k^2 K''(0) + \dots$. If write in the spatial space, we have

$$S[\phi] = \int d^d x (c_2 (\partial\phi)^2 + c_0 \phi^2 + c_4 \phi^4) \quad (35)$$

This is the so-called ϕ -4 theory or action. It is a phenomenological theory of Ising model.

Also, this function is also called the Ginzburg-Landau functional. It differs from Landau expansion of the Gibbs free energy Eq. 4 by the first gradient term (apart from the more general notation).

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