Notes on the BCS theory

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Contents

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In this chapter, we will consider superconductivity and related BCS theory. The era of superconductivity started in 1911. Yet there was no microscopic understanding of the effect until 1957 when Bardeen, Cooper and Schrieffer produced what still remains the theory of superconductors.

First, let us briefly review the experimental features. There are two main features: zero resistance and Meissner effect. Zero resistance in the transport measurement is easy to understand. In 1933 Meissner and Ochsenfeld discovered that weak magnetic fields are expelled by superconductors. This perfect diamagnetism, now known as the Meissner effect. If a sufficiently strong magnetic field is applied, the superconductivity is partially or completely destroyed. In so-called type-I superconductors, fields exceeding the thermodynamic critical field $H_c(T)$ completely destroy the superconductivity. Type-II superconductors behave somewhat differently. They also exhibit perfect diamagnetism up to a critical field $H_{c1}(T)$, known as the lower critical field. Above this field the magnetization decreases continuously rather than suddenly. Magnetic flux begins to penetrate the sample (producing what is known as the "mixed state"), but does not destroy the superconducting state until the so-called upper critical field $H_{c2}(T)$ has been reached. The way that the sample excludes the magnetic flux is to generate currents at the surface which produce an opposing magnetic flux. The flux density does not fall to zero discontinuously inside the superconductor since this would require infinite current density on the surface. Rather the magnetic flux density goes to zero exponentially with a characteristic length λ known as the penetration depth.

Next we can outline several key points to understand the physics of superconductivity from the theoretical viewpoint. The first step is, two electrons form the so-called cooper pair, driving by some kind of attractive interaction. Then, a superconductor can roughly be described as a superfluid (roughly a Bose-Einstein condensate) of cooper pairs (charge-2e bosons).

COOPER PAIR

Given a very weak attractive potential in D dimensions, does there exist a bound state? Let us explore this question, following the study of Cooper in 1956.

Suppose we start with a Fermi sea and we add two electrons. These two electrons can only be added above the fermi wavevector k_F. The important point is we consider the attractive

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interaction between two electrons:

$$H_{int} = -V \sum_{q,k_1,k_2,\sigma_1,\sigma_2} c^+_{k_1+q,\sigma_1} c^+_{k_2-q,\sigma_2} c_{k_2,\sigma_2} c_{k_1,\sigma_1} \tag{1}$$

The appearance of attractive interaction is highly nontrivial. We will discuss that later.

For simplicity, we take two spin antiparallel to each other, and their momentum are opposite to each other ($q = 2k_F$). The hamiltonian should be

$$H_{eff} = \sum_k E_k (c^+_k c_k + c^+_{-k} c_{-k}) - V \sum_{k,k'>k_F} c^+_{k'} c^+_{-k'} c_{-k} c_k \tag{2}$$

Let us write an appropriate trial bound state for the two added electrons

$$\psi = \sum_{k>k_F} a(k) c^+_k c^+_{-k} |F> \tag{3}$$

where $|F>$ is the filled Fermi surface state.

The energy of these two electrons is

$$E[a] = 2 \sum_{k>k_F} E_k |a(k)|^2 - V \sum_{k,k'>k_F} a^*(k') a(k) \tag{4}$$

To minimize $a(k)$ gives the lowest energy E. Since the wave function should be normalized, we need to introduce this condition to the energy functional:

$$E[a] = 2 \sum_{k>k_F} E_k |a(k)|^2 - V \sum_{k,k'>k_F} a^*(k') a(k) + \lambda(1 - \sum_k |a(k)|^2) \tag{5}$$

$$\frac{\delta E[a]}{\delta a} = 0 \Rightarrow (2E_k - \lambda) a(k) = V \sum_{k'>k_F} a(k') \tag{6}$$

This equation gives

$$a(k) = \frac{V \sum_{k'>k_F} a(k')}{2E_k - \lambda} \tag{7}$$

$$\Rightarrow 1 = V \sum_{k>k_F} \frac{1}{2E_k - \lambda} \tag{8}$$

For $V > 0$ (related to the attractive interaction), there is a solution for λ.

We take some approximations here, see below.

$$\sum_{k>k_F} \frac{1}{2E_k - \lambda} \approx \int_0^\Lambda dE \frac{D(E)}{2E - \lambda} \approx D(E_F) \int_0^\Lambda dE \frac{1}{2E - \lambda} = D(E_F) \ln \frac{2\Lambda - \lambda}{-\lambda} \tag{9}$$

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so we have

$$1 = VD(E_F)\ln\frac{2\Lambda-\lambda}{-\lambda} \Rightarrow \lambda = \frac{-2\Lambda}{e^{\frac{1}{D(E_F)V}}-1} \tag{10}$$

Next from $(2E_k-\lambda)a(k) = V\sum_{k'>k_F} a(k')$, we obtain

$$\lambda = 2\sum_k E_k|a(k)|^2 - V\sum_{k,k'>k_F} a^*(k)a(k') = E \tag{11}$$

So we have

$$E = \frac{-2\Lambda}{e^{\frac{1}{D(E_F)V}}-1} \overset{D(E_F)V\ll 1}{=} -2\Lambda\exp[-\frac{1}{D(E)V}] < 0 \tag{12}$$

In conclusion, in the presence of a Fermi sea an arbitrarily weak attractive potential will form a bound state! The key here is that the Pauli exclusion principle facilitates electron binding.

Here there are several remarks. 1) No matter how weak the interaction is, the bound state is formed. 2) We assume the the fermi surface is stable, and only consider two electrons above the Fermi level. But this conclusion implies that the Fermi surface is not stable, it has instability toward superconductivity. 3) $\exp(-\frac{1}{D(E_F)V})$ means that this is not an perturbative solution. One cannot solve this problem within perturbation calculation.

MECHANISM OF ATTRACTION

In the strongly-correlated systems, the replusive interaction is usual but the attractive interaction is rare. The reason is discussed in the Chap 1, where the Coulomb interaction between electrons always replusive (they have the same electric charge).

We can think about other mechanism for attractive interaction. The electron-phonon interaction is one of the possibility. Electrons coupled with lattice vibration could give an attractive interaction. And the net interaction, combining both electron-electron replusive interaction and electron-phonon attractive interaction, could be attractive in some limit. The detailed analysis will be left for the future studies.

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Let us write a simple model of electrons and phonons

$$\begin{aligned} H &= H_0 + H_1 \\ H_0 &= \sum_{\mathbf{k}} E_{\mathbf{k}} c^+_{\mathbf{k}} c_{\mathbf{k}} + \sum_{\mathbf{q}} \omega_{\mathbf{q}} a^+_{\mathbf{k}} a_{\mathbf{k}} \\ H_1 &= \sum_{\mathbf{q},\mathbf{k}} [M c^+_{\mathbf{k}+q} c_{\mathbf{k}} a^+_{\mathbf{q}} + h.c.] \end{aligned} \tag{13}$$

where H_0 is the Hamiltonian for uncoupled phonons and electrons and H_1 includes the small coupling between the two. Here we assume Einstein phonons (with polarization indices suppressed). M is an interaction matrix element between electron and phonon and we will take it as a constant for simplicity. The physics here is that any charge density that builds up is able to couple to phonons.

The Hamiltonian $H = H_0 + H_1$ is a complicated interacting Hamiltonian, and is generally difficult to solve. However, we can take advantage of the fact that the coupling constant M is small. This enables us to work perturbatively in this parameter. Our idea is to remove the phonons from consideration and determine what the effective interaction is between electrons. This is known as "integrating out" the phonons. To do this we make a canonical transformation on our Hamiltoninan

$$H_{eff} = e^{-S} H e^{S} = H + [H, S] + \frac{1}{2}[[H, S], S] + ... \tag{14}$$

The idea is to choose S so that the electron part of the Hamiltonian becomes completely decoupled from the phonons (at the price of introducing an effective interaction between the electrons). We can make this decoupling order by order in the small parameter M, where H_1 is order M and we assume that S is also order M (since if M is zero, we are decoupled with $S = 0$ also). Let us rewrite the series, order by order in the small parameter

$$H_{eff} = (H_0) + (H_1 + [H_0, S]) + ([H_1, S] + \frac{1}{2}[[H_0, S], S]) + ... \tag{15}$$

where each set of terms in parenthesis () are of the same order. Let us choose S so as to eliminate the first order term

$$H_1 + [H_0, S] = 0 \rightarrow \langle n|H_1|m\rangle = -\langle n|[H_0, S]|m\rangle = -(E_n - E_m)\langle n|S|m\rangle \tag{16}$$

Plugging this back into the expansion we get

$$H_{eff} = H_0 + \frac{1}{2}[H_1, S] + ... \tag{17}$$

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We have thus eliminated the coupling between electrons and phonons at first order, leaving only the second order term.

We now want to figure out what H_{eff} does to make an effective interaction between electrons. Let us consider two states $|a(b)\rangle$ without phonons excited and the transited intermediated state $|c\rangle$ with one phonon excitation:

$$|a\rangle = c^+_{k-q} c^+_{p+q}|FS\rangle, \quad |b\rangle = c^+_k c^+_p|FS\rangle, \quad |c\rangle = c^+_{k-q} c^+_p a^+_q|FS\rangle \tag{18}$$

We are interested in finding the effective interaction matrix element

$$\begin{aligned}
\frac{1}{2}\langle a|[H_1, S]|b\rangle &= \frac{1}{2}\sum_c [\langle a|H_1|c\rangle\langle c|S|b\rangle - \langle a|S|c\rangle\langle c|H_1|b\rangle] \\
&= \frac{1}{2}\sum_c \langle a|H_1|c\rangle\langle c|H_1|b\rangle[\frac{1}{E_b - E_c} + \frac{1}{E_a - E_c}] \\
&\sim \frac{1}{2}\sum_c \langle a|H_1|c\rangle\langle c|H_1|b\rangle[\frac{1}{E_k - \hbar\omega_q - E_{k-q}} + \frac{1}{E_{p+q} - \hbar\omega_q - E_p}] \\
&\sim -\sum_c \langle a|H_1|c\rangle\langle c|H_1|b\rangle\frac{1}{\hbar\omega_q}
\end{aligned} \tag{19}$$

where we assume the energies are

$$E_c = \hbar\omega_q + E_{k-q} + E_p, E_b = E_k + E_p, E_a = E_{k-q} + E_{p+q} \tag{20}$$

Now if the energy differences between the fermion states are small compared to $\hbar\omega!$ then the interaction is attractive. Writtern in the operator form is

$$H_{eff} = -\frac{|M|^2}{\hbar\omega}\sum_{k,k',q} c^+_{k+q} c^+_{k'-q} c_{k'} c_k \tag{21}$$

THE BCS THEORY

We have discussed that the Cooper pair will form due to the attractive interaction. Next we can assume the order parameter of superconducitivity is non-zero:

$$\Delta_k \equiv \langle 0|c^+_k c^+_{-k}|0\rangle \neq 0 \tag{22}$$

Next we will use the mean-field method again:

$$c^+_k c^+_{-k} = \langle c^+_k c^+_{-k}\rangle + (c^+_k c^+_{-k} - \langle c^+_k c^+_{-k}\rangle) \tag{23}$$

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So the mean-field hamiltonian becomes ($\Delta = \sum_k \Delta_k$)

$$\begin{aligned} H &\approx \sum_k E_k(c_k^+ c_k + c_{-k}^+ c_{-k}) - V\sum_{k,k'}[c_{-k}c_k\langle c_k^+ c_{-k}^+\rangle + c_{k'}^+ c_{-k'}^+\langle c_{-k}c_k\rangle - \langle c_{-k}c_k\rangle\langle c_k^+ c_{-k}^+\rangle] \\ &= \sum_k E_k(c_k^+ c_k + c_{-k}^+ c_{-k}) - \Delta\sum_k (c_k^+ c_{-k}^+ + c_{-k}c_k) + \Delta^2/V \end{aligned} \tag{24}$$

This model can be solved by the Bogoliubov transformation. And the result is

$$H = E_s(0) + \sum_k \xi_k(\alpha_k^+\alpha_k + \alpha_{-k}^+\alpha_{-k}) \tag{25}$$

$$E_s(0) = 2\sum_k E_k v_k^2 - 2\Delta\sum_k u_k v_k + \Delta^2/V = \sum_k (E_k - \xi_k) + \Delta^2/V \tag{26}$$

where $u_k^2 = \frac{1}{2}(1+\frac{E_k}{\xi_k}), v_k^2 = \frac{1}{2}(1-\frac{E_k}{\xi_k})$, $\xi_k = \sqrt{\Delta^2 + E_k^2}$ and

$$\alpha_k^+ = u_k c_k^+ - v_k c_{-k} \tag{27}$$

$$\alpha_k = u_k c_k - v_k c_{-k}^+ \tag{28}$$

is the (Bogoliubov) quasiparticle creation and annihilation operator. To see its physical meaning, we can check the case of $\Delta = 0$,

$$u_k^2 = \frac{1}{2}(1+\frac{E_k}{\xi_k}) = \begin{cases} 1, k > k_F \\ 0, k < k_F \end{cases}, v_k^2 = \frac{1}{2}(1-\frac{E_k}{\xi_k}) = \begin{cases} 0, k > k_F \\ 1, k < k_F \end{cases} \tag{29}$$

is just the Fermi distribution for electron (u_k) and hole (v_k). So we have α_k^+ creates particles like

$$\alpha_k^+ = u_k c_k^+ - v_k c_{-k} = \begin{cases} c_k^+, k > k_F(\texttt{create an electron below Fermi level}) \\ c_{-k}, k < k_F(\texttt{create a hole below Fermi level}) \end{cases}, \tag{30}$$

$$\alpha_k = u_k c_k - v_k c_{-k}^+ = \begin{cases} c_k, k > k_F(\texttt{annihilate an electron above Fermi level}) \\ c_{-k}^+, k < k_F(\texttt{annihilate a hole below Fermi level}) \end{cases} \tag{31}$$

For the case of $\Delta \neq 0$, u_k, v_k are nonzero above and below the Fermi level. α^+, α represents a combination of electron and hole.

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`Bogoliubov Transformation`

In theoretical physics, the Bogoliubov transformation, named after Nikolay Bogolyubov, is a unitary transformation from a unitary representation of some canonical commutation relation algebra or canonical anticommutation relation algebra into another unitary representation, induced by an isomorphism of the commutation relation algebra. The Bogoliubov transformation is often used to diagonalize Hamiltonians, which yields the steady-state solutions of the corresponding Schrodinger equation. The solutions of BCS theory in a homogeneous system, for example, are found using a Bogoliubov transformation. The Bogoliubov transformation is also important for understanding the Unruh effect, Hawking radiation, pairing effects in nuclear physics, and many other topics.

Boson.— Consider the canonical commutation relation for bosonic creation and annihilation operators in the harmonic basis

$$[\hat{a}, \hat{a}^\dagger] = 1 \tag{32}$$

Define a new pair of operators

$$\hat{b} = u\hat{a} + v\hat{a}^\dagger \tag{33}$$

$$\hat{b}^\dagger = u^*\hat{a}^\dagger + v^*\hat{a} \tag{34}$$

where the latter is the hermitian conjugate of the first.

The Bogoliubov transformation is the canonical transformation mapping the operators $\hat{a}$ and $\hat{a}^\dagger$ to $\hat{b}$ and $\hat{b}^\dagger$. To find the conditions on the constants u and v such that the transformation is canonical, the commutator is evaluated,

$$[\hat{b}, \hat{b}^\dagger] = [u\hat{a} + v\hat{a}^\dagger, u^*\hat{a}^\dagger v^*\hat{a}] = (|u|^2 - |v|^2)[\hat{a}, \hat{a}^\dagger] = 1 \tag{35}$$

It is then evident that $|u|^2 - |v|^2 = 1$ is the condition for which the transformation is canonical. We can choose $u = e^{i\phi_1} cosh\theta$ and $v = e^{i\phi_2} sinh\theta$.

Fermion.— For the anticommutation relation,

$$\{\hat{a}, \hat{a}^\dagger\} = 1 \tag{36}$$

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the same transformation with u and v becomes

$$\{\hat{b}, \hat{b}^\dagger]\} = \{u\hat{a} + v\hat{a}^\dagger, u^*\hat{a}^\dagger v^*\hat{a}\} = (|u|^2 + |v|^2)\{\hat{a}, \hat{a}^\dagger\} = 1 \tag{37}$$

To make the transformation canonical, u and v can be parameterized as $u = e^{i\phi_1}cos\theta$ and $v = e^{i\phi_2}sin\theta$

`Application to the BCS-type Hamiltonian`

Consider for fermion operators the Hamiltonian

$$H = \epsilon(c_1^\dagger c_1 + c_2^\dagger c_2) + (\lambda c_1^\dagger c_2^\dagger + \lambda^* c_2 c_1) \tag{38}$$

which arises in the BCS theory of superconductivity. Note that the opposite ordering of labels in the terms $c_1^\dagger c_2^\dagger$ and $c_2 c_1$, which is also a requirement of Hermiticity.

The fermionic Bogoliubov transformation is

$$\begin{aligned} c_1^\dagger &= u d_1^\dagger + v d_2 \\ c_2^\dagger &= u d_2^\dagger - v d_1 \end{aligned}$$

where u and v are c-numbers. The requirement of commutation relation is

$$\{c_1^\dagger, c_1\} = \{u d_1^\dagger + v d_2, u^* d_1 + v^* d_2^\dagger\} = |u|^2\{d_1^\dagger, d_1\} + |v|^2\{d_2, d_2^\dagger\} = |u|^2 + |v|^2 = 1 \tag{39}$$

and so we must require $|u|^2 + |v|^2 = 1$, suggesting the parameter $u = e^{i\phi/2}\cos\theta$, $v = e^{-i\phi/2}\sin\theta$.

The remaining step is to substitute H for $c^\dagger$ and c in terms of $d^\dagger$ and d, and pick θ so that terms in $d_1^\dagger d_2^\dagger + d_2 d_1$ have vanishing coefficient. The calculation is clearest when it is set out using matrix notation. First, we can write H as

$$H = \frac{1}{2}\begin{pmatrix} c_1^\dagger & c_2 & c_2^\dagger & c_1 \end{pmatrix} \begin{pmatrix} \epsilon & \lambda & 0 & 0 \\ \lambda^* & -\epsilon & 0 & 0 \\ 0 & 0 & \epsilon & -\lambda \\ 0 & 0 & -\lambda^* & -\epsilon \end{pmatrix} \begin{pmatrix} c_1 \\ c_2^\dagger \\ c_2 \\ c_1^\dagger \end{pmatrix} + \epsilon \tag{40}$$

where we have used the anticommutator to make substitutions of the type $c^\dagger c = 1 - cc^\dagger$.

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For conciseness, consider just the upper block

$$
\begin{aligned}
&\begin{pmatrix} c_1^\dagger & c_2 \end{pmatrix} \begin{pmatrix} \epsilon & \lambda \\ \lambda^* & -\epsilon \end{pmatrix} \begin{pmatrix} c_1 \\ c_2^\dagger \end{pmatrix} \\
&= \begin{pmatrix} d_1^\dagger & d_2 \end{pmatrix} \begin{pmatrix} u & -v^* \\ v & u^* \end{pmatrix} \begin{pmatrix} \epsilon & \lambda \\ \lambda^* & -\epsilon \end{pmatrix} \begin{pmatrix} u^* & v^* \\ -v & u \end{pmatrix} \begin{pmatrix} d_1 \\ d_2^\dagger \end{pmatrix} \\
&= \begin{pmatrix} \epsilon|u|^2 - \lambda uv - \lambda^* u^* v^* - \epsilon|v|^2 & \epsilon uv^* + \lambda u^2 - \lambda^*(v^*)^2 + \epsilon uv^* \\ \epsilon u^* v + \lambda^*(u^*)^2 - \lambda v^2 + \epsilon u^* v & -\epsilon|u|^2 + \lambda uv + \lambda^* u^* v^* + \epsilon|v|^2 \end{pmatrix}
\end{aligned}
$$

We require that

$$\epsilon uv^* + \lambda u^2 - \lambda^*(v^*)^2 + \epsilon uv^* = 0 \tag{41}$$

$$\epsilon|u|^2 - \lambda uv - \lambda^* u^* v^* - \epsilon|v|^2 = \tilde{\epsilon} \tag{42}$$

We need some algothrim to get θ:

$$
\begin{aligned}
&\epsilon \cos\theta \sin\theta e^{i\phi} + \lambda \cos^2\theta e^{i\phi} - \lambda^* \sin^2\theta e^{i\phi} + \epsilon \cos\theta \sin\theta e^{i\phi} = 0 \\
&\epsilon \cos\theta \sin\theta + \lambda \cos^2\theta - \lambda^* \sin^2 + \epsilon \cos\theta \sin\theta = 0 \\
&\Longrightarrow \tan 2\theta = -\lambda/\epsilon
\end{aligned}
$$

thus

$$u^2 = \frac{1}{2}\left(-\frac{\epsilon}{\sqrt{\epsilon^2+\lambda^2}} + 1\right), v^2 = \frac{1}{2}\left(\frac{\epsilon}{\sqrt{\epsilon^2+\lambda^2}} + 1\right) \tag{43}$$

and

$$\tilde{\epsilon} = -\sqrt{\epsilon^2+\lambda^2} \tag{44}$$

Including the other 2×2 block of H, we conclude that

$$H = \tilde{\epsilon}(d_1^\dagger d_1 + d_2^\dagger d_2) - \tilde{\epsilon} + \epsilon \tag{45}$$

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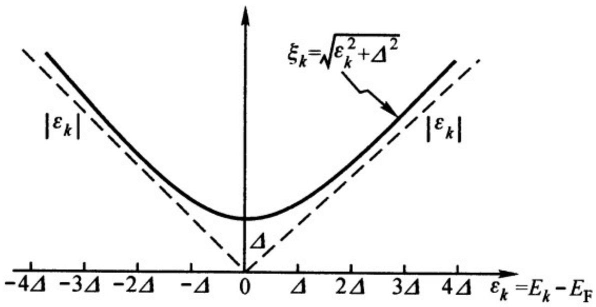

FIG. 1: The energy dispersion of Bogoliubov quasiparticles.

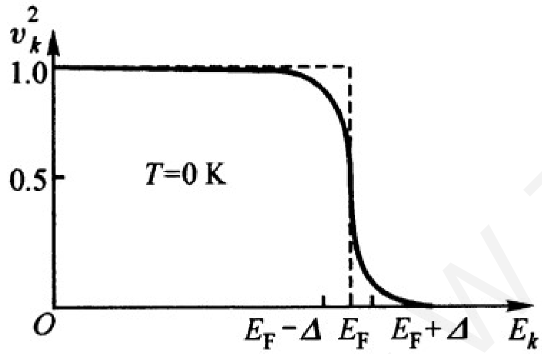

FIG. 2: The function v_k^2 as a function energy.

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Quasiparticle energy gap

The quasiparticle energy excitation gap is

$$\Delta = V\sum_k \langle c_{-k}c_k\rangle$$
$$= V\sum_k \langle (u_k\alpha_{-k} - v_k\alpha_k^+)(u_k\alpha_k + v_k\alpha_{-k}^+)\rangle = V\sum_k u_k v_k = \frac{V}{2}\sum_k \frac{\Delta}{\xi_k} \tag{46}$$

There are two solutions: One is trivial $\Delta = 0$, the other one is

$$1 = \frac{V}{2}\sum_k \frac{1}{\sqrt{E_k^2+\Delta^2}} = V\int_0^\Lambda \frac{D(E)dE}{\sqrt{E^2+\Delta^2}} \tag{47}$$

$$\Rightarrow \Delta \approx 2\Lambda \exp[-\frac{1}{D(0)V}] \tag{48}$$

BCS ground state wave function

The ground state wave function is written as ($c_k|vac >= 0$ where $|vac >$ is the vaccum state for normal particles)

$$\Pi_k \alpha_k \alpha_{-k}|vac\rangle = \Pi_k(u_k v_k + v_k^2 c_k^+ c_{-k}^+)|vac > \tag{49}$$

This state satisfies with the condition that it vanishes when apply α_k:

$$\alpha_p \Pi_k \alpha_k \alpha_{-k}|vac\rangle = (\alpha_p \alpha_p)\alpha_{-p}\Pi_{k\neq p}\alpha_k \alpha_{-k}|vac\rangle = 0 \tag{50}$$

since we have $\alpha_k \alpha_k = 0$ (ferminic relation $\{\alpha_k, \alpha_k\} = 0$).

Then we deal with the normalization:

$$\begin{aligned} &< vac|\Pi_k(u_k v_k + v_k^2 c_{-k} c_k)(u_k v_k + v_k^2 c_k^+ c_{-k}^+)|vac > \\ &= \Pi_k(u_k^2 v_k^2 + v_k^4) = \Pi_k v_k^2 \end{aligned} \tag{51}$$

So we define the BCS wavefunction as

$$|0 >= \frac{1}{\Pi_k v_k}\Pi_k(u_k v_k + v_k^2 c_k^+ c_{-k}^+)|vac >= \Pi_k(u_k + v_k c_k^+ c_{-k}^+)|vac > \tag{52}$$

Using the wave function, we can calculate the physical observations. For example, for the density of electrons:

$$\langle 0|c_k^+ c_k|0\rangle = \langle 0|(u_k \alpha_k^+ + v_k \alpha_{-k})(u_k \alpha_k + v_k \alpha_{-k}^+)|0\rangle = v_k^2 \langle 0|\alpha_{-k}\alpha_{-k}^+|0\rangle = v_k^2 \tag{53}$$

So v_k^2 is the occupation probability of electrons. It drops from 1 to 0 around the Fermi surface.

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Supercurrent

Previous discussion only focus on the static case, i.e. the total momentum of Cooper pairs are zero: $\mathbf{K} = \mathbf{k} + (-\mathbf{k}) = 0$. Next we consider the non-equilibrium case just a little bit away from the static case. Say we consider a small external electric field drifts the momentum of Cooper pair to be $\mathbf{k} + \delta\mathbf{k}$ and $-\mathbf{k} + \delta\mathbf{k}$, so the total momentum is $2\delta\mathbf{k}$. This will produce supercurrent, with current density

$$j_s = \frac{n}{2}(-2e)\frac{2\hbar\delta k}{2m} = -\frac{ne\hbar\delta k}{m} \tag{54}$$

Since this current is in superfluid phase, it flows without any dispassion. If the external field is dropped, this supercurrent doesnot reduce, which persists in the superconducting systems.

But the strength of supercurrent has a upper limit. Taking the kinetic energy of this supercurrent as

$$\delta E \approx \frac{n\hbar^2(\delta k)^2}{2m} \tag{55}$$

If this kinetic energy is larger than the condensation energy, $\sim \frac{1}{2}D(0)\Delta^2$, the superconducting state is not stable. So we use this energy scale to estimate the upper limit of supercurrent,

$$|\delta k_c| = \sqrt{\frac{mD(0)}{n}}\frac{\Delta}{\hbar} \to j_s^c = \frac{ne\hbar}{m}\delta k_c \propto \Delta \tag{56}$$

So the supercurrent directly depends on superconducting gap. The physical meaning is that, if the energy is larger than Δ, the electron and hole quasiparticles will be formed, which leads to finite resistivity and dispassion, killing the superconducting state.

Finite-temperature calculation

The direct measurement targeting the superconducting state in experiment is critical temperature T_c, instead of Δ. So we need to calculate T_c. For $T > 0$, we need to reconsider the gap equatoin as

$$\begin{aligned}
\Delta(T) &= V\sum_k \langle c_{-k}c_k\rangle = V\sum_k \langle (u_k\alpha_{-k} - v_k\alpha_k^+)(u_k\alpha_k + v_k\alpha_{-k}^+)\rangle \\
&= V\sum_k u_kv_k(\langle \alpha_{-k}\alpha_{-k}^+\rangle - \langle \alpha_k^+\alpha_k\rangle) = V\sum_k u_kv_k(1 - \langle \alpha_{-k}^+\alpha_{-k}^+\rangle - \langle \alpha_k^+\alpha_k\rangle) \\
&= V\sum_k u_kv_k(1 - 2f(\xi_k))
\end{aligned} \tag{57}$$

where $\xi_k = \sqrt{E_k^2 + \Delta^2(T)}$. It leads to

$$1 = \frac{V}{2}\sum_k \frac{1 - 2f(\xi_k)}{\xi_k} = \frac{V}{2}\sum_k \frac{\tanh\frac{\beta\xi_k}{2}}{\xi_k} \tag{58}$$

Solving this equation self-consistently gives the dependence of gap $\Delta(T)$, and the critical temperature T_c can be obtained also.

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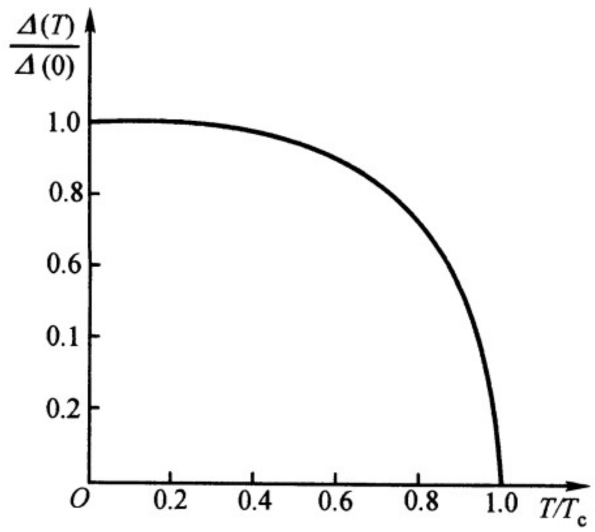

FIG. 3: The dependence of superconducting gap on the temperature.

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Experimental evidence

At last, how to verify the BCS theory in experiment? For conventional superconductors (which includes the vast majority of supercondutors known), fortunately, there is quite a bit of good evidence that it is the coupling to the phonons that is crucial. A very strong piece of evidence comes from the so-called "isotope effect". One considers different isotopes of the same material, i.e., adding and subtracting a few neutrons from the nucleus of atoms. The neutrons do nothing to the electronic properties of the material, or the band structure. The only thing they do is to change the frequency of vibration of the atoms by changing their masses. In these conventional superconductors it is usually found that the critical temperature for superconductivity scales as $T_c \sim M^{-\alpha}$, where M is the mass of the atomic nucleus, where $\alpha \approx 0.5$. The fact that the nuclear mass plays any role in the superconductivity is a sure sign that phonons are crucial to the mechanism.

JOEPHEON JUNCTION

In earlier sections we have argued that superconductivity results from the existence of a condensate. The Josephson effect is a remarkable manifestation of the existence of such a condensate. Instead of developing a fully microscopic theory, in this section we present only the minimal phenomenology needed to understand the Josephson effect.

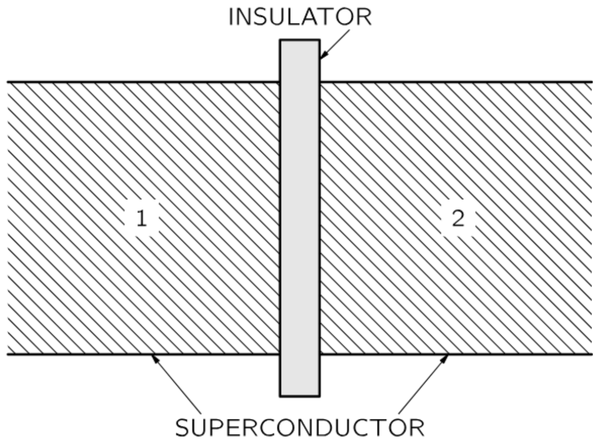

FIG. 4: A insulating layer separating two superconductors.

In order to analyze such a junction I'll call the amplitude to find an electron on one side, ψ_1, and the amplitude to find it on the other, ψ_2. In the superconducting state the wave function ψ_1 is the common wave function of all the electrons on one side, and ψ_2 is the corresponding function on the other side. I could do this problem for different kinds of superconductors, but let us take a very simple situation in which the material is the same on both sides so that the junction is symmetrical and simple. Also, for a moment let there be no magnetic field. Then the two amplitudes should be related in the following way:

$$i\frac{\partial\psi_1}{\partial t} = U_1\psi_1 + K\psi_2$$
$$i\frac{\partial\psi_2}{\partial t} = U_2\psi_2 + K\psi_1 \tag{59}$$

The constant K is a characteristic of the junction, which is the coupling between the two sides by the amplitude K that there may be leakage from one side to the other. Suppose that we connect the two superconducting regions to the two terminals of a battery so that there is a potential difference V across the junction: $U_1 - U_2 = qV$.

Importantly, we would write the condensate wavefunction as

$$\psi_1 = \sqrt{n_1}e^{i\phi_1}, \psi_2 = \sqrt{n_2}e^{i\phi_2} \tag{60}$$

which leads to

$$\dot{\sqrt{n_1}} + i\dot{\sqrt{n_1}} = -i(eV\sqrt{n_1} + K\sqrt{n_2}e^{i(\phi_1-\phi_2)}) \tag{61}$$

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Using the combination of complex conjugation, we have

$$2\dot{\sqrt{n_1}} = -iK(\sqrt{n_2}e^{i(\phi_1-\phi_2)} - \sqrt{n_2}e^{i(\phi_2-\phi_1)}) = K\sqrt{n_2}2\sin(\phi_2-\phi_1) \tag{62}$$

$$\rightarrow \dot{n_1} = 2K\sqrt{n_1 n_2}\sin(\phi_2-\phi_1) \tag{63}$$

Similarly we solve the n_2 dependence as

$$\dot{n_2} = -2K\sqrt{n_1 n_2}\sin(\phi_2-\phi_1) \tag{64}$$

And we can solve the phase function as

$$\dot{\phi_1} = -(eV + K\sqrt{n_2/n_1}\cos(\phi)) \tag{65}$$

$$\dot{\phi_2} = (eV - K\sqrt{n_1/n_2}\cos(\phi)) \tag{66}$$

Since the current is proportional to the change of charge, we have the AC josephson current as (if $n_1 = n_2$)

$$I(t) = I_c\sin(\phi(t)), \dot{\phi(t)} \approx 2eV(t) \tag{67}$$

THE BDG EQUATION

In the earlier sections, our study of superconductivity has mostly been based on the BCS reduced Hamiltonian, which is formulated in momentum space, and relies heavily on translation symmetry. Translation symmetry is sometimes broken, for example by the presence of disorder, or (which is of particular interest) a magnetic field, and this renders the momentum-space formulation much less useful. In this section we develop a formalism (often associated with the names Bogoliubov and de Gennes, or BdG) that allows us to study pairing and superconductivity in real space. Although this formalism was originally formulated in the continuum, here we focus on lattice models instead for their simplicity, and the continuum version can be reached by taking the continuum limit of a tight-binding lattice.

Let us start with the hubbard model with attractive on-site interaction which could favor the s-wave pairing:

$$H = \sum_{i\sigma,j\sigma'} h_{i\sigma,j\sigma'}c^\dagger_{i,\sigma}c_{j,\sigma'} - U\sum_i n_{i,\uparrow}n_{i,\downarrow} \tag{68}$$

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where fermion operators $c_{i\sigma}$ satisfies the relationship

$$\{c_{i\sigma}, c^\dagger_{j\sigma'}\} = \delta_{ij}\delta_{\sigma\sigma'}, \quad \{c_{i\sigma}, c_{j\sigma'}\} = 0 = \{c^\dagger_{i\sigma}, c^\dagger_{j\sigma'}\}$$

The interaction term is treated within mean-field level:

$$\begin{aligned}
-U\sum_i n_{i,\uparrow}n_{i,\downarrow} &= -Uc^\dagger_{i,\uparrow}c_{i,\uparrow}c^\dagger_{i,\downarrow}c_{i,\downarrow} = -Uc^\dagger_{i,\downarrow}c^\dagger_{i,\uparrow}c_{i,\uparrow}c_{i,\downarrow} \\
&= -U[\Delta_{ii} + (c_{i,\uparrow}c_{i,\downarrow} - \Delta_{ii})][\Delta^*_{ii} + (c^\dagger_{i,\downarrow}c^\dagger_{i,\uparrow} - \Delta^*_{ii})] \\
&\approx -U[|\Delta_{ii}|^2 + (c_{i,\uparrow}c_{i,\downarrow} - \Delta_{ii})\Delta^*_{ii} + \Delta_{ii}(c^\dagger_{i,\downarrow}c^\dagger_{i,\uparrow} - \Delta^*_{ii})] \\
&= U|\Delta_{ii}|^2 - U\Delta^*_{ii}c_{i,\uparrow}c_{i,\downarrow} - U\Delta_{ii}c^\dagger_{i,\downarrow}c^\dagger_{i,\uparrow}
\end{aligned}$$

where we define the order parameter

$$\Delta_{ii} = \langle c_{i,\uparrow}c_{i,\downarrow}\rangle, \qquad \Delta^*_{ii} = \langle c^\dagger_{i,\downarrow}c^\dagger_{i,\uparrow}\rangle$$

The hamiltonian becomes

$$H = \sum_{i\sigma,j\sigma'} h_{i\sigma,j\sigma'}c^\dagger_{i,\sigma}c_{j,\sigma'} - U\sum_i[\Delta^*_{ii}c_{i,\uparrow}c_{i,\downarrow} + \Delta_{ii}c^\dagger_{i,\downarrow}c^\dagger_{i,\uparrow}] + U\sum_i|\Delta_{ii}|^2 \tag{69}$$

Next we get

$$\begin{aligned}
[c_{i,\uparrow}, H] &= \sum_{j,\sigma'} h_{i\uparrow,j\sigma'}c_{j,\sigma'} + U\Delta_{ii}c^\dagger_{i,\downarrow} \\
[c_{i,\downarrow}, H] &= \sum_{j,\sigma'} h_{i\downarrow,j\sigma'}c_{j,\sigma'} - U\Delta_{ii}c^\dagger_{i,\uparrow} \\
[c^\dagger_{j,\uparrow}, H] &= -\sum_{i,\sigma} h_{i\sigma,j\uparrow}c^\dagger_{i,\sigma} - U\Delta^*_{ii}c_{i,\downarrow} \\
[c^\dagger_{j,\downarrow}, H] &= -\sum_{i,\sigma} h_{i\sigma,j\downarrow}c^\dagger_{i,\sigma} + U\Delta^*_{ii}c_{i,\uparrow}
\end{aligned}$$

With the help of Bogoliubov transformation

$$\hat{c}_{i\sigma} = \sum_n[u^n_{i\sigma}\hat{\gamma}_n - \sigma v^{n*}_{i\sigma}\hat{\gamma}^\dagger_n] \tag{70}$$

$$\hat{c}^\dagger_{i\sigma} = \sum_n[u^{n*}_{i\sigma}\hat{\gamma}^\dagger_n - \sigma v^n_{i\sigma}\hat{\gamma}_n] \tag{71}$$

, the hamiltonian becomes quadrutic term:

$$H = \sum_n E_n\gamma^\dagger_n\gamma_n \tag{72}$$

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By inserting the Bogoliubov transformation into Eq. , we get

$$
\begin{aligned}
u^n_{i\uparrow}E_n &= \sum_{j,\sigma'} h_{i\uparrow,j\sigma'}u^n_{j\sigma'} + U\Delta_{ii}v^n_{i,\downarrow} \\
v^{n*}_{i,\uparrow}E_n &= \sum_{j,\sigma'} h_{i\uparrow,j\sigma'}(-\sigma')v^{n*}_{j,\sigma'} + U\Delta_{ii}u^{n*}_{i,\downarrow} \\
u^n_{i\downarrow}E_n &= \sum_{j,\sigma'} h_{i\downarrow,j\sigma'}u^n_{j,\sigma'} + U\Delta_{ii}v^n_{i,\uparrow} \\
v^{n*}_{i,\downarrow}E_n &= \sum_{j,\sigma'} h_{i\downarrow,j\sigma'}(\sigma')v^{n*}_{j,\sigma'} + U\Delta_{ii}u^{n*}_{i,\uparrow}
\end{aligned}
$$

After solving the above equations, the self-consistency condition is calculated as

$$
\begin{aligned}
\Delta_{ii} &= \langle c_{i,\uparrow}c_{i,\downarrow}\rangle = \langle [\sum_n (u^n_{i\uparrow}\hat{\gamma}_n - v^{n*}_{i\uparrow}\hat{\gamma}^\dagger_n)][\sum_m (u^m_{i\downarrow}\hat{\gamma}_m + v^{m*}_{i\downarrow}\hat{\gamma}^\dagger_m)]\rangle \\
&= \langle \sum_{m,n} u^n_{i\uparrow}u^m_{i\downarrow}\hat{\gamma}_n\hat{\gamma}_m - v^{n*}_{i\uparrow}v^{m*}_{i\downarrow}\hat{\gamma}^\dagger_n\hat{\gamma}^\dagger_m + u^n_{i\uparrow}v^{m*}_{i\downarrow}\hat{\gamma}_n\hat{\gamma}^\dagger_m - v^{n*}_{i\uparrow}u^m_{i\downarrow}\hat{\gamma}^\dagger_n\hat{\gamma}_m\rangle \\
&= \sum_n u^n_{i\uparrow}v^{n*}_{i\downarrow}\langle\hat{\gamma}_n\hat{\gamma}^\dagger_n\rangle - v^{n*}_{i\uparrow}u^n_{i\downarrow}\langle\hat{\gamma}^\dagger_n\hat{\gamma}_n\rangle \\
&= \sum_n u^n_{i\uparrow}v^{n*}_{i\downarrow}(1-f(E_n)) - v^{n*}_{i\uparrow}u^n_{i\downarrow}f(E_n) \qquad (73) \\
\Delta_{ii} &= -\langle c_{i,\downarrow}c_{i,\uparrow}\rangle = -\langle [\sum_m (u^m_{i\downarrow}\hat{\gamma}_m + v^{m*}_{i\downarrow}\hat{\gamma}^\dagger_m)][\sum_n (u^n_{i\uparrow}\hat{\gamma}_n - v^{n*}_{i\uparrow}\hat{\gamma}^\dagger_n)]\rangle \\
&= -\sum_n [u^n_{i\uparrow}v^{n*}_{i\downarrow}\langle\hat{\gamma}^\dagger_n\hat{\gamma}_n\rangle - v^{n*}_{i\uparrow}u^n_{i\downarrow}\langle\hat{\gamma}_n\hat{\gamma}^\dagger_n\rangle] \\
&= -\sum_n [u^n_{i\uparrow}v^{n*}_{i\downarrow}f(E_n) - v^{n*}_{i\uparrow}u^n_{i\downarrow}(1-f(E_n))] \qquad (74) \\
\Rightarrow \Delta_{ii} &= \frac{1}{2}\sum_n [u^n_{i\uparrow}v^{n*}_{i\downarrow} + u^n_{i\downarrow}v^{n*}_{i\uparrow}]\tanh(\frac{\beta E_n}{2}) \qquad (75)
\end{aligned}
$$

where the Fermion-Dirac distribution is

$$
f(E_n) = \frac{1}{e^{\beta E_n}+1}, \quad f(-E_n) = 1 - f(E_n), \quad 1 - 2f(E_n) = \tanh(\frac{\beta E_n}{2}) \qquad (76)
$$

For density operator,

$$
\begin{aligned}
n_{i\sigma} &= \langle c^\dagger_{i\sigma}c_{i\sigma}\rangle = \langle [\sum_n (u^{n*}_{i\sigma}\hat{\gamma}^\dagger_n - \sigma v^n_{i\sigma}\hat{\gamma}_n)][\sum_m (u^m_{i\sigma}\hat{\gamma}_m - \sigma v^{m*}_{i\sigma}\hat{\gamma}^\dagger_m)]\rangle \\
&= \sum_n |u^n_{i\sigma}|^2 f(E_n) + |v^n_{i\sigma}|^2 f(-E_n) \qquad (77)
\end{aligned}
$$

SUMMARY

Here we try to introduce the microscopic theory of BCS superconductor and related BCS theory. In a very short way, you can just take superconductor as "charged-2e superfluid".

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I have to say, there are so many topics missing here, say Ginzberg-Landau effective theory, London equation, Messiner effect, and Phase effecto of superconductor. It leaves for the readers. In future, we will consider the superconducting theory from the perspective of field theory.

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HOMEWORK

1. Calculate the condensation energy, due to the condensation of cooper pairs in a normal metal.

[1] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. 108, 1175 (1957)

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