Lecture note for the Anderson localization and metal-to-insulator transition

W. Zhu[1]

[1] *Westlake Institute of Advanced Study,*
Westlake University, Hangzhou, P. C. China

Contents

Keywords: Localization; Anderson insulator; Scaling theory; Quantum transport and Landauer formula; Lyapunov exponent

The effect of impurity atoms on the properties of solids is of considerable physical interest, because most real materials contain impurities which affect their physical properties. In this chapter, we consider electron scattering by impurity atoms. We assume the electrons to be non-interacting, but is perturbed by the potential due to the impurity atom. This problem can immediately be seen to be expressible in terms of the quantum mechanics of scattering of a single electron from the impurity potential.

One important phenomenon induced by the disordered impurity scattering is Anderson localization, which was first proposed by P. W. Anderson in 1957 in his seminal paper, "*On the absence of diffusion in certain random lattices*". We will cover this topic here. We start with the Drude formula for conductivity: $\sigma = e^2 D\rho$, where D is the diffusion constant, ρ is the electronic density of states (DOS) per unit volume at the Fermi level for non-interacting electrons at zero temperature. Thus far, in the band theory, the only insulators we have encountered are band insulators, in which $\rho = 0$ because the Fermi energy lies inside a band gap, and as a result the DOS is zero. These are systems like diamond or sodium chloride which have a gap between the highest fully occupied band (the valence band) and the lowest unoccupied band (the conduction band). The DOS is zero in this gap, and the system cannot respond (dissipatively) to weak electric fields. It can polarize (i.e. it has a dielectric constant greater than unity), but it cannot carry current at zero temperature where there are no thermally excited carriers. Hence the origin of band-insulator physics lies in simple single-electron physics and the formation of (filled) bands due to the periodic crystalline structure. Next, we are now going to encounter a new type of insulator called the Anderson insulator, in which $\rho > 0$, but $D = 0$ due to the fact that electron eigenstates at the Fermi level have been localized by the random impurity potential, so that the charges can no longer diffuse in the long-time or large distance limit. These systems are metals in the absence of disorder and they have a finite DOS at the Fermi level. As disorder is increased, the conductivity falls and finally goes to zero at some critical value of the disorder, at which point a metal–insulator phase transition occurs. The density of states at the Fermi energy remains finite and yet the system cannot carry dc current. As we will discuss in detail, the origin of this is quantum interference, which is missing in the Boltzmann transport theory.

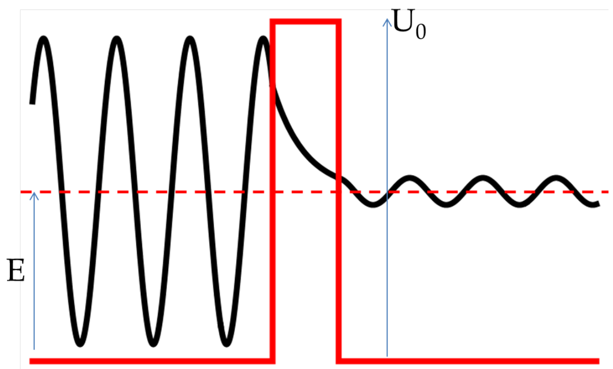

FIG. 1: Potential barrier.

INTUITION FROM THE QUANTUM MECHANICS

Consider a potential barrier, as represented in Fig. 1 . The potential is constant V_0 between $x = 0$ and $x = a$, and zero outside of this region. A particle starts on one side of the barrier, and we want to know the possibility of it crossing to the other side of the barrier. First, we study the particle with energy $E < V_0$, so that classically it has no chance of crossing the barrier. This is the strong disorder regime in the following discussion. Then, we discuss the $E > V_0$, relating to the weak disorder regime.

Tunneling in a 1D Potential Barrier $E < V_0$

Because of the simple nature of this potential, we can in fact solve the Schrödinger equation analytically for this setup

$$H\psi = -\frac{\hbar^2}{2m}\nabla^2\psi + V(x)\psi = E\psi \tag{1}$$

The wave function takes the form of

$$\psi(x) = \begin{cases} e^{ikx} + Re^{-ikx}, x < 0 \\ Ae^{\kappa x} + Be^{-\kappa x}, 0 < x < a \\ Te^{ikx}, x > a \end{cases} \tag{2}$$

Again we assume a beam of definite momentum incident from the left and no wave incident from the right. For the solutions outside the barrier,

$$k = \sqrt{\frac{E}{2m}} \tag{3}$$

Inside the barrier

$$\kappa = \sqrt{2m(V_0 - E)/\hbar^2} \tag{4}$$

Using the continuum condition at $x = 0$:

$$1 + R = A + B \tag{5}$$

$$ik/\kappa(1 - R) = A - B \tag{6}$$

at $x = a$:

$$Ae^{\kappa a} + Be^{-\kappa a} = Te^{ika} \tag{7}$$

$$Ae^{\kappa a} - Be^{-\kappa a} = (ik/\kappa)Te^{ika} \tag{8}$$

Using these conditions we have

$$T = e^{-ika} \frac{-2ik/\kappa}{[1 - (k/\kappa)^2]\sinh(\kappa a) - 2i(k/\kappa)\cosh(\kappa a)} \tag{9}$$

So the transmission probability is:

$$\begin{aligned} |T|^2 &= \frac{4k^2\kappa^2}{(k^2 - \kappa^2)^2\sinh^2(\kappa a) + 4k^2\kappa^2\cosh^2(\kappa a)} = \frac{4k^2\kappa^2}{(k^2 + \kappa^2)^2\sinh^2(\kappa a) + 4k^2\kappa^2} \\ &= \frac{1}{1 + \frac{1}{\frac{E}{V_0}(1 - \frac{E}{V_0})}\sinh^2(\kappa a)} \end{aligned} \tag{10}$$

which should be less than ≤ 1. In the case of $\kappa a \ll 1$ (strong scattering limit), we have

$$|T|^2 \approx \frac{16k^2\kappa^2}{(k^2 + \kappa^2)} e^{-2\kappa a} \tag{11}$$

which is exponetially small.

Tunneling in a 1D Potential Barrier $E > V_0$

We can do an analytically continuation,

$$\kappa \to ik', k' = \sqrt{2m(E - V_0)\hbar^2} \tag{12}$$

and

$$\begin{aligned} |T|^2 &= \frac{4k^2(k')^2}{(k^2 + (k')^2)^2\sin^2(k'a) + 4k^2(k')^2} \\ &= \frac{1}{1 + \frac{1}{4}(\frac{k}{k'} - \frac{k'}{k})^2\sin^2(k'a)} \end{aligned} \tag{13}$$

which is also smaller than 1. Interestingly, one may find that, if $\sin(k'a) = 0$ (resonance condition), the transmission probability is 1. Just in this case the travelling wave exists so the wave is extended in the whole system.

Intuition

- For $E < V_0$, the transmission probability of a single barrier decay always exponentially. Think about there are many barriers, the transmission probability should be decayed exponentially too.
- For $E > V_0$, the transmission probability $|T|^2 < 1$ for most of cases. Think about n such potentials with random distribution (V_0 is random and a is also random, so $\sin(k'a) = 0$ cannot hold for all potentails), the total transmission should be like $|T|^{2n} \ll 1$.
- For mixed cases with potential barriers and wells, one simply expects $|T|^2 \ll 1$.

Through this analysis, we expect that, in the presence of random (on-site) potentials, the total transmission is

$$T(E) = T_1(E)T_2(E)...T_N(E) \to 0, \forall E \tag{14}$$

That is, all states with energy E are localized state. This is a simple intuition for the Anderson localization.

TRANSFER-MATRIX IN THE 1D LATTICE

A direct calculation

We consider a 1D lattice,

$$Ea_n = \epsilon_n a_n + a_{n+1} + a_{n-1} \tag{15}$$

where a_n is the wave function amplitude on local site n. It can be wriiten in the form of transfer matrix

$$\begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix} = \begin{pmatrix} E - \epsilon_n & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_n \\ a_{n-1} \end{pmatrix} \equiv P_n \begin{pmatrix} a_n \\ a_{n-1} \end{pmatrix} = \prod_n P_n \begin{pmatrix} a_1 \\ a_0 \end{pmatrix} \tag{16}$$

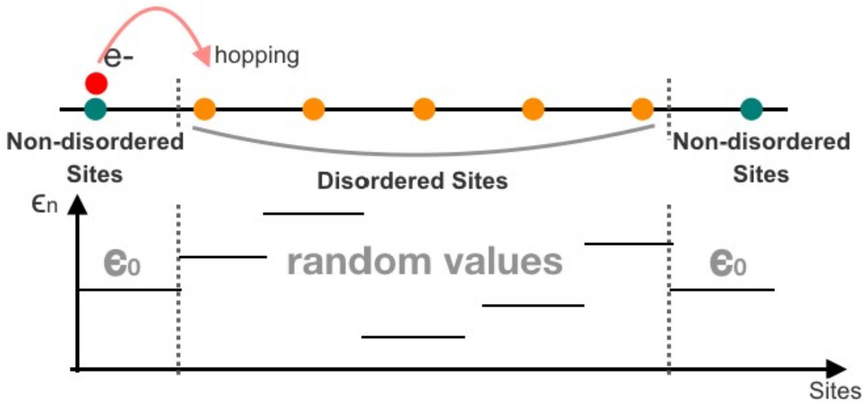

FIG. 2: Disordered lattice and non-disordered boundary condition.

The elements of the transfer matrix P_n^{ij} satisfy the recursion relations as follows:

$$P_n^{11} = (E-\epsilon_n)P_{n-1}^{11} - P_{n-2}^{11} \tag{17}$$

$$P_n^{21} = P_{n-1}^{11} \tag{18}$$

$$P_n^{12} = (E-\epsilon_n)P_{n-1}^{12} - P_{n-2}^{12} \tag{19}$$

$$P_n^{22} = P_{n-1}^{12} \tag{20}$$

Making the squared and average, say

$$\langle (P_n^{11})^2\rangle = \langle (E-\epsilon_n)^2 (P_{n-1}^{11})^2 + (P_{n-2}^{11})^2 - 2(E-\epsilon_n)P_{n-1}^{11}P_{n-2}^{11}\rangle \tag{21}$$

$$= \langle (E-\epsilon_n)^2 (P_{n-1}^{11})^2\rangle + \langle (P_{n-2}^{11})^2\rangle \tag{22}$$

similar with $(P^{12})^2$. So we will have the similar transfer matrix for averaged matrix

$$\begin{pmatrix} \langle (P_{n+1}^{11})^2\rangle \\ \langle (P_n^{11})^2\rangle \end{pmatrix} = \begin{pmatrix} \langle (E-\epsilon_n)^2\rangle & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \langle (P_n^{11})^2\rangle \\ \langle (P_{n-1}^{11})^2\rangle \end{pmatrix} \tag{23}$$

$$\begin{pmatrix} \langle (P_{n+1}^{12})^2\rangle \\ \langle (P_n^{12})^2\rangle \end{pmatrix} = \begin{pmatrix} \langle (E-\epsilon_n)^2\rangle & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \langle (P_n^{12})^2\rangle \\ \langle (P_{n-1}^{12})^2\rangle \end{pmatrix} \tag{24}$$

This averaged transfer matrix has the eigenvalues like $\lambda_\pm$:

$$\lambda_\pm = \frac{\langle (E-\epsilon_n)^2\rangle}{2} \pm \sqrt{1 + (\frac{\langle (E-\epsilon_n)^2\rangle}{2})^2} \tag{25}$$

Please note that $\lambda_+ > 1$.

Next we can relate two boundaries of the random lattice using matrix T and matrix S:

$$S\begin{pmatrix} A \\ D \end{pmatrix} = \begin{pmatrix} r & t \\ t & r \end{pmatrix}\begin{pmatrix} A \\ D \end{pmatrix} = \begin{pmatrix} B \\ C \end{pmatrix} \tag{26}$$

$$T\begin{pmatrix} B \\ A \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}\begin{pmatrix} B \\ A \end{pmatrix} = \begin{pmatrix} D \\ C \end{pmatrix} \tag{27}$$

where A is input wave amplitude, B is reflected amplitude, C is transmission amplitude. Relating S and T, we have

$$r = S_{11} = -\frac{T_{12}}{T_{11}} \tag{28}$$

$$t = S_{21} = \frac{1}{T_{11}} \tag{29}$$

$$\tag{30}$$

$R + T = r^2 + t^2 = 1.$

Next we consider the quantum transport of resistivity. In quantum transport, resistivity reflects how difficult the wave propagates. To quantify it, we can write down the following formula: $\rho \equiv \frac{R}{T}$, which is determined by transmission and reflection wave amplitude, then you can check that ρ takes values from 0 to ∞. Using this formula, the resistivity relates to the T-matrix as

$$\rho = \frac{R}{T} = (-T_{12}^{*})(-T_{12}) = |T_{12}|^2 \tag{31}$$

We connect the disorder region with non-disorder regime (see Fig. 2). We assume the wave is just plane waves in the non-disorder regime:

$$a(n) = Ae^{ikn} + Be^{-ikn}, n \leq 1 \tag{32}$$

$$a(n) = Ce^{ikn} + De^{-ikn}, n \geq N \tag{33}$$

Then we use the T-matrix to connect with the non-disorder regime

$$\begin{pmatrix} a_1 \\ a_0 \end{pmatrix} = \begin{pmatrix} e^{-ikd} & e^{ikd} \\ 1 & 1 \end{pmatrix}\begin{pmatrix} B \\ A \end{pmatrix} = \Lambda\begin{pmatrix} B \\ A \end{pmatrix} \tag{34}$$

$$\begin{pmatrix} a_{N+1} \\ a_N \end{pmatrix} = \begin{pmatrix} e^{-ikd} & e^{ikd} \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^{-ikNd} & 0 \\ 0 & e^{ikNd} \end{pmatrix}\begin{pmatrix} D \\ C \end{pmatrix} = \Lambda\theta^{-1}\begin{pmatrix} D \\ C \end{pmatrix} \tag{35}$$

So we have

$$\Lambda\theta^{-1}\begin{pmatrix} D \\ C \end{pmatrix} = [\prod_n P_n]\Lambda \begin{pmatrix} B \\ A \end{pmatrix}, \quad \Rightarrow T = \theta\Lambda^{-1}[\prod_n P_n]\Lambda \tag{36}$$

$$T_{12} = P_N^{22} e^{-idk(N-1)} + P_N^{12} e^{ikd(N+1)} + P_N^{21} e^{idkN} + P_N^{11} e^{ikd(N+2)} \tag{37}$$

Using this result, we can calculate the resistivity as

$$\langle\rho\rangle \sim \langle (P_N^{11})^2\rangle + \langle (P_N^{21})^2\rangle + \langle (P_N^{12})^2\rangle + \langle (P_N^{22})^2\rangle \sim \lambda_+^N + \lambda_-^N \sim e^{N\ln\lambda_+} \tag{38}$$

Since $\lambda_+ > 1$, $\langle\rho\rangle$ grows exponentially with N, leading to the localization of wave in 1D.

Now we are ready to make formal definition as below. In a 1D disordered chain, electrons experience Anderson localization, which requires the wave function takes the form of

$$|\varphi(x)| \sim e^{-x/\xi_{loc}}. \tag{39}$$

The localization length ξ_{loc} characterizes the exponential decay of wavefunctions. Equivalently, the Lyapunov exponent γ quantifies the exponential divergence of trajectories in a disordered system and is related to the localization length by:

$$\gamma = \frac{1}{2\xi_{loc}} \tag{40}$$

and the resistivity is

$$\rho \sim e^{2\gamma N} \tag{41}$$

The Lyapunov exponent measures the exponential rate of separation (or convergence) of infinitesimally close trajectories in a system. It is widely studied in dynamical systems and disordered systems.

- $\gamma > 0$: Chaotic/disordered system (trajectories diverge exponentially).
- $\gamma = 0$: Neutral stability (e.g., periodic motion).
- $\gamma < 0$: Stable fixed point (trajectories converge).

In the dynamic systems, Lyapunov exponent can quatify the chaos. For example, let us consider the Lorenz attractor (weather model) or logistic map $x_{n+1} = rx_n(1 - x_n)$. For $r > 3.57$, the logistic map exhibits chaos with Lyapunov exponent $\gamma > 0$. Lyapunov exponent quantifies the “butterfly effect”: Small errors in initial conditions grow exponentially, limiting predictability.

Landauer-Büttiker Formalism.— In quantum transport, the conductance G of a 1D channel is given by the Landauer formula:

$$G = \frac{2e^2}{h} T, \tag{42}$$

where T is the transmission probability.

Here, we explain the Landauer Formalism as following. The current I through a 1D conductor with transmission probability $T(E)$ is given by:

$$I = \frac{2e}{h} \int_{-\infty}^{\infty} T(E) \left[f_L(E) - f_R(E) \right] dE,$$

where $f_L(E)$ and $f_R(E)$ are the Fermi-Dirac distributions of the left and right reservoirs, e is the electron charge, h is Planck's constant.

For a perfect conductor ($T = 1$) at zero temperature and small bias voltage V. The Fermi-Dirac distributions reduce to step functions: $f_L(E) - f_R(E) = 1$ in the energy window $\mu_R < E < \mu_L$, The chemical potential difference is $\mu_L - \mu_R = eV$, where V is the applied voltage. The current simplifies to:

$$I = \frac{2e}{h} \cdot T \cdot eV.$$

This gives the conductance $G = I/V$.

The total resistance R_{total} includes contributions from both the scatterer and the contacts:

$$R_{total} = \frac{h}{2e^2} (\frac{1}{T} - 1) = \frac{h}{2e^2} \frac{R}{T}. \tag{43}$$

It is worth noting, we introduce a contact resistance $R_c = \frac{h}{2e^2}$ in the Landauer-Büttiker formalism arises from the quantum mechanical mismatch between a macroscopic reservoir (e.g., a metal contact) and a 1D conductor (e.g., a nanowire or quantum point contact). Even a perfect conductor (with transmission $T = 1$) exhibits this resistance because of the finite number of quantum channels available to carry current.

If geometric factors (length LL, cross-sectional area AA) are normalized, the resistivity simplifies to:

$$\rho = \frac{R}{T} \tag{44}$$

This relationship captures how quantum coherence and scattering govern resistivity in nanoscale systems.

One can also solve transfer matrix numerically. Fig. 3 shows the correlation length from the transfer matrix calculation, by applying 10^6 transfer matrices to an initial vector $(a(1), a(0)) = (1; \exp(ik))$, where k is the momentum of a tightly bound electron with a given energy E. The resulting coefficients $a(n)$ are averaged from 100 calculations with different realizations of randomness. We then extract the localization length ξ by exponentially fitting the coefficients. The disorder dependence of the localization is well described by the power law $\xi \sim W^{-2}$. Moreover, states at the edges of the energy band localize fastest.

A formal prove

A mathematical proof of Anderson localization in 1D has been achieved. The idea is product of random transfer matrix satisfies the Fuerstenberg's theorem [see H. Furstenberg and H. Kesten, The Annals of Mathematical Statistics 31, 457 (1960); and K. Ishii, Progress of Theoretical Physics Supplement 53, 77 (1973)]. By proving Fuerstenberg's theorem for products of random matrices, it can be rigorously shown that solutions $a(n)$ decay exponentially, so that the overall shape of the wavefunction is $a(n) \sim \exp(-L/\xi)$. Here, ξ is the localization length in units of lattice constant a. As a consequence, eigenstates are not extended over real-space but are exponentially confined to certain areas of the lattice. We will refer to these states as localized. Remarkably, we note that any state in a one dimensional system will be localized (for random potential). This implies that any 1D material will lose its diffusion or conduction capabilities if the system size is $L > \xi$.

THOULESS PICTURE OF CONDUCTANCE

In classical physics, and more specifically the Drude and (semiclassical) Boltzmann transport theories, electric transport is characterized by the conductivity σ, which is a local property of a conductor (the local value of the current is linearly proportional to the local value of the electric field). Electric transport is better characterized by the conductance G, which can have a highly non-trivial dependence on the linear size (or length scale) L of the conductor being studied. Such scale dependence reveals the nature of electronic states.

Assuming Ohm's law and the Einstein relation do apply (at a certain scale), we have the

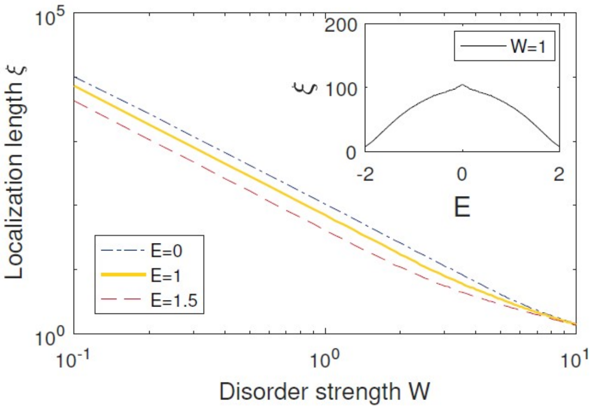

FIG. 3: Log-log scale plot of localization length ξ in 1D as a function of disorder W and energy E (inset). The data was numerically calculated by solving the transfermatrix method. ξ is well fitted by $\xi = W^{-2}$ for all E.

conductance of a cubic-shaped conductor in d dimensions

$$G(L) = \sigma L^{d-2} = e^2 D \frac{dn}{d\mu} L^{d-2} = e^2 \frac{D}{L^2} \frac{d(L^d n)}{d\mu} = \frac{2e^2}{\Delta t \Delta E} \tag{45}$$

where

$$\Delta t = L^2/D \tag{46}$$

is the time it takes an electron to diffuse over a distance (of order) L, and

$$\Delta E = (\frac{d(L^d n/2)}{d\mu})^{-1} = (\frac{dN(E)}{dE})^{-1}|_{E_F} \tag{47}$$

is the average energy-level spacing at the Fermi energy of this finite-size conductor ($N(E)$ is number of states below energy E for each spin).

Thouless defined an energy scale, from the Heisenberg uncertainty relation we expect a broadening of the energy level at the Fermi energy of order

$$E_T = \frac{h}{\Delta t} \tag{48}$$

known as the Thouless energy. Thus we have

$$G(L) = \frac{2e^2}{h} \frac{E_T}{\Delta E} \equiv \frac{2e^2}{h} g_T(L) \tag{49}$$

In the Thouless picture, g_T is the ratio between the energy-level uncertainty and the energy-level spacing of the conductor at the Fermi level.

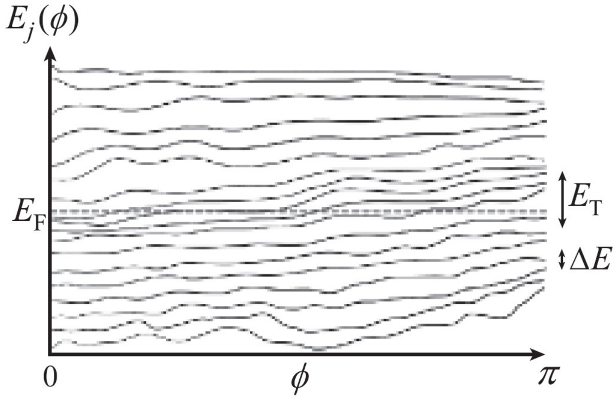

FIG. 4: Schematic band structure for a quasi-1D metal constructed from periodically repeated blocks. The average level spacing is ΔE. For a good metal we expect that the Thouless energy E_T greatly exceeds the level spacing: $E_T \gg \Delta E$. If one follows a single level from the zone center to the zone boundary, the typical change in energy of that level is E_T. Since this greatly exceeds the level spacing and since (generically) the levels do not cross, it must be the case that the motion of nearby levels is correlated. The range of correlation covers g_T levels. Hence this is typical number of levels crossing the Fermi energy.

In theoretical (and, in particular, numerical) studies of conductors, we often consider isolated systems in order to simplify the physical setting. In an isolated and finite-size system, the energy levels are discrete and sharp. Is it possible to estimate E_T in such studies? Thouless further observed that leads are coupled to the conductor at its boundaries and can be viewed as a boundary effect; thus E_T is related to the sensitivity of the energy level to a change of the boundary condition. Thouless thus suggested estimating ET as the change of energy for the state (labeled i) at the Fermi level when the boundary condition changes from periodic to antiperiodic along one direction (while sticking to a periodic boundary condition in the other directions):

$$E_T \approx E_i(\phi = \pi) - E_i(\phi = 0) \tag{50}$$

where ϕ is the boundary-condition angle for the wave function in the x direction:

$$\psi(x + L) = e^{i\phi}\psi(x) \tag{51}$$

Given $g(L)$, the question now is, what is the (dimensionless) conductance $g(2L)$? One might expect to be able to express the dimensionless conductance at scale 2L in the form

$$g(2L) = F[\frac{E_T(L)}{\Delta E(L)}] = F[g(L)] \tag{52}$$

in terms of some (unknown) scaling function F that relates the conductance of a piece of conductor with size L to that of a piece of size $2L$.

SCALING THEORY FOR LOCALIZATION

We would like to finish our introduction to random impurities by touching on the concept of electron localization. In fact, disorder actually gives rise to collective interference effects within the electron gas, which ultimately lead to the localization of the electron wavefunction. This idea was proposed in 1958 by Philip W. Anderson, and subsequently named after him. Our modern understanding of electron localization was greatly aided by a conceptual breakthrough on this problem made by Don Liciardello and David Thouless, who proposed that the resistance of a material, or rather the inverse resistance, the conductance $G = 1/R$, is a function of scale. Thouless' idea, initially proposed for one dimension, was taken up by the "Gang of Four", Elihu Abrahams, Philip W. Anderson, Don Licciardello, and Tirupattur Ramakrishnan, and extended to higher dimensions, leading to the modern scaling theory of localization.

Stimulated by the results in two dimensions, Abrahams et al. were led to propose that, in any dimension, conductance or inverse resistance $G = 1/R$ could always be normalized to form a dimensionless parameter

$$g(L) = \frac{G(L)}{e^2/h} \tag{53}$$

which satisfies a one-parameter scaling equation:

$$\frac{d \ln g(L)}{d \ln L} = \beta(g) \tag{54}$$

When conductance is large we expect semi-classical approach in metal is valid, so we may use the Drude model, so that $g(L) = \frac{ne^2\tau}{m} L^{d-2}$ and

$$\beta(g) = d - 2, g \to \infty \tag{55}$$

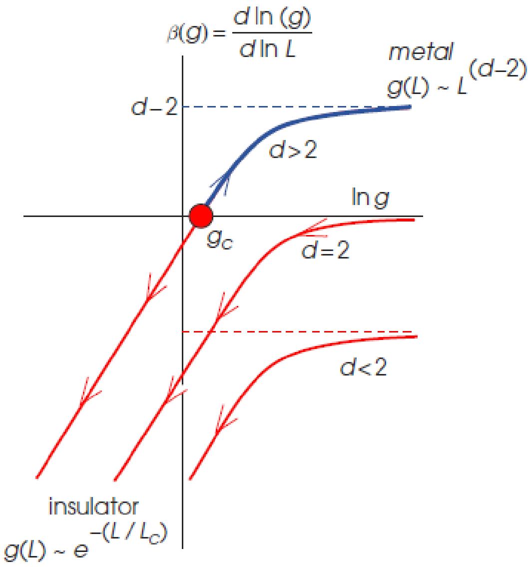

FIG. 5: The scaling function $\beta(g)$ deduced by Abrahams et al. for a non-interacting metal. For $d > 2$ there is critical conductance g_c which gives rise to a disorder-driven metal–insulator transition. For $d \leq 2$ disorder always gives rise to localization and the formation of an insulator.

When the conductance was small $g \to 0$ on scales longer than the localization length L_c, they argued that $g(L)$ would decay exponentially, $g(L) \approx \exp(-L/\xi)$, so

$$\beta(g) \sim \ln g, g \to 0 \tag{56}$$

By connecting these two asymptotic limits, "Gang of Four" reasoned that the beta function for conductance would take the form shown in Figure 5. In dimensions $d \leq 2$, $\beta(g)$ is always negative, so the conductance always scales to zero and electrons are always localized. However, in dimensions $d > 2$ there is a disorder-driven metal–insulator transition at the critical conductance $g = g_c$. As the amount of disorder is increased when the short-distance conductance g passes below g_c, the material becomes an insulator in the thermodynamic limit.

In 3D, g_c is usually non-zero, which leads to the "minimal conductance" as first proposed by Mott (Mott, 1975), i.e. the electrically conducting materials all have 3D conductivities larger than a certain minimum value.

In 3D, near g_c, we discuss the critical behavior

$$\beta(g) \sim s[\ln g - \ln g_c] = s \ln \frac{g}{g_c} \tag{57}$$

where s is the slope of the β function when it crosses zero at $\ln g_c$. From the definition

$$\frac{d\ln g}{\ln\frac{g}{g_c}} = s d\ln L \overset{\texttt{integration}}{\rightarrow} \frac{\ln(g/g_c)}{\ln g_0/g_c} = (L/l)^s \tag{58}$$

where g_0 is the value of g at $L = l$. Let us define $\epsilon = \ln g_0 - \ln g_c$. For $\epsilon \gg 1$, we have $\beta = d - 2 = 1 = s\ln(g/g_c)$ (by definition),

$$\ln\frac{g}{g_c} = \epsilon(L/l)^s \rightarrow s^{-1} = \epsilon(\zeta/l)^s \rightarrow \zeta \sim l(s\epsilon)^{-1/s} \tag{59}$$

Therefore, in 3D there is a length scale ζ, dubbed as correlation length, which defines the property of wave field in the extended regime. Close to the mobility edge $\epsilon \rightarrow 0$, we see $\zeta \rightarrow \infty$.

In the case of $\beta < 0$ or $\epsilon < 0$,

$$\ln\frac{g}{g_c} = -|\epsilon|(L/l)^s \rightarrow \ln\frac{g}{g_c} = -(L/(l|\epsilon|^{-1/s}))^s \approx -L/\xi, \tag{60}$$

because we know in the limit of $L \rightarrow \infty$, $\beta \propto \ln g \rightarrow s = 1$ (by assumption). So localization length ξ is determined by

$$\xi \sim l|\epsilon|^{-1/s} \sim l|\epsilon|^{-1} \tag{61}$$

which is divergent near the mobility edge (criticality).

The scaling law shown in Figure 5 can be verified numerically.

APPLICATIONS

Ioffe-Regel limit of bad metal

Let us write the resistivity as:

$$\rho = \frac{m}{ne^2\tau} = \frac{mv_F}{ne^2 l} \sim \frac{1}{D(E_F)l} \tag{62}$$

So we derive the condition for bad metal with large resistivity:

$$D(E_F)l \leq 1 \tag{63}$$

This is called Ioffe-Regel condition for bad metal. According to this criterion, the localization tends to occur near the band edge which relates to small DoS. The mobility edge is defined as the boundary between the localized states and the extended states.

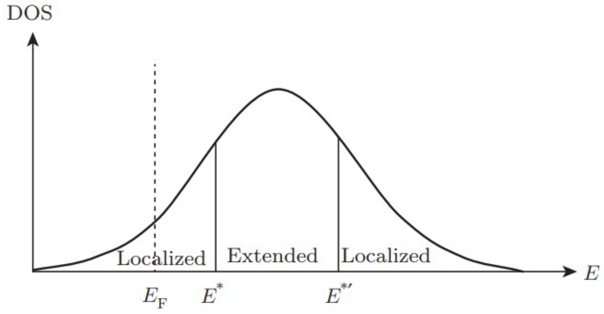

FIG. 6: A sketch of the DOS and mobility edges of the 3D Anderson model with $W < W_c$. Localized states occur near the band edges, while extended states exist near the band center (unless all states are localized by strong disorder, which is the case for $W > W_c$).

Hopping conductivity

We consider an inelastic hopping process between two localized states. Suppose that we consider hopping between two states whose center positions are separated by a distance R. Mott argued that the typical matrix element squared for a hop of distance R will be of order $e^{-R/\xi}$ due to the exponentially small overlap of the localized states.

Suppose without loss of generality that the energy splitting of the two states is $\delta\epsilon = E_2 - E_1 > 0$. We will first assume that the lower energy state is occupied by an electron and that the upper one is empty. If $k_B T \ll \delta\epsilon$, the density of phonons of the required energy $\delta\epsilon$ will be very low. This will contribute a factor $n_B(\delta E) \sim e^{-\beta\delta\epsilon}$ so that the transition rate will be proportional to

$$\Gamma \sim e^{-\beta\delta\epsilon} e^{-R/\xi} \sim \exp[-\frac{1}{k_B T(dn/d\mu)R^d} - \frac{R}{\xi}] \tag{64}$$

The last equation comes from, at low temperatures the system will strongly prefer to hop between states with the smallest possible value of $\delta\epsilon$. Let us recall that the typical level spacing in a box of size R is $\delta\epsilon \sim ((dn/d\mu)R^d)^{-1}$. Thus small δE typically requires searching a large volume $\sim R^d$ to find a state close by in energy.

This varies extremely strongly with distance and we may safely assume that the hopping rate is dominated by hops whose distance R minimizes the exponent:

$$\frac{\partial \ln \Gamma}{\partial R} = \frac{dR^{-(d+1)}}{k_B T dn/d\mu} - \frac{1}{\xi} = 0 \Rightarrow \frac{R}{\xi} \sim (\frac{T_0}{T})^{\frac{1}{d+1}} \tag{65}$$

where $T_0 = (k_B \xi^d (dn/d\mu)/d)^{-1}$ s an estimate of the characteristic energy scale associated with the localization length. This clearly demonstrates that for $T \ll T_0$ the typical hopping distance greatly exceeds the localization length and so our assumption is self-consistent.

At the optimal distance R, we have

$$\Gamma \sim e^{-a(\frac{T_0}{T})^{\frac{1}{d+1}}} \tag{66}$$

As the temperature is lowered, the typical hop distance grows larger and larger yielding "stretched exponential".

MANY-BODY LOCALIZATION

In the previous part, we have neglected electron- electron interactions. This approximation is only reasonable for systems that behave ideally. For an improved theoretical description, we will add an on-site repulsion term to the Anderson Hamiltonian and obtain the Bose-Hubbard Hamiltonian with random on-site potential

$$H = W\sum_i \epsilon_i c_i^\dagger c_i + t\sum_{ij}(c_i^\dagger c_j + h.c.) + U\sum_i n_i(n_i - 1) \tag{67}$$

With including interaction effect, how about the localization problem? So far, if or not the many-body version of localization, dubbed many-body localization, can survive in systems with repulsive interaction is still under debate.

Eigenstate Thermalization Hypothesis

We consider an isolated quantum system that we partition in parts A and B. Originally, this system is prepared in state

$$|\psi\rangle = \sum_{ab} c_{ab}|A\rangle \otimes |B\rangle \tag{68}$$

We can readily write down the density matrix for sub- system A:

$$\rho_A = Tr_B[|\psi\rangle\langle\psi|] \tag{69}$$

The Eigenstate Thermalization Hypothesis now states [J. M. Deutsch, Physical Review A 43, 2046 (1991). M. Srednicki, Physical Review E 50, 888 (1994).], that if

Condition 1) the diagonal terms of ρ_A are a smooth function of energy E of state

Condition 2) the off-diagonal terms decohere (i.e. interfere destructively since they are the sum of many oscillating terms of different frequencies)

Then, the reduced density matrix will be equal to a statistical density matrix (i.e. canonical) after some euqilibration time. We can write this as

$$\rho_A = \rho_A^{th} = exp(-\beta H_A) \tag{70}$$

The effective temperature β in the canonical description is obtained by requiring that the expectation values of the energy are the same: $\langle H\rangle = Tr[\rho^{th}H]$. Simply put, this means: A small isolated quantum system may act as its own bath and thermalize. We note however, that this hypothesis - if true - is of great benefit. The numerical calculation of density matrices for isolated quantum systems is only possible for fairly small system sizes while the statistical treatment does not require much effort for arbitrary dimension of the Hilbert-space.

Many-body localization

The modern question of many-body localization (MBL) is essentially the following. If we have an interacting system which is isolated (e.g. a cold atomic gas in an isolated optical lattice not coupled to an external bath that can cause dephasing), are the states at finite energy density (above the ground state) localized? Loosely speaking, do the inelastic-scattering processes associated with the interactions cause the system to act as its own (dephasing) bath? This is a complex question which is still under debate and remains an open question. Nevertheless, at the time of writing, I think the answer is negative: MBL is not reasonable even in one-dimension.

Whether or not the appearance of MBL is an even more subtle question than for single-particle Anderson localization. As already noted, the dimension of the Hilbert space is exponentially larger for the MBL problem, and estimating the distance (in Hilbert space rather than in real space) between nearly degenerate many-body states and the perturbation-theory matrix elements that connect them is not trivial. However, the basic physical picture is one of self-consistency: if MBL occurs, the system is frozen and unable to act as its own bath to dephase and mix states. In this case the locally conserved operators are adiabatically connected to the bare spin operators (and have support only over other spin operators on nearby sites). One possibility is, there exists a set of conservation laws, and particles/spins do not diffuse under the interacting Hamiltonian, and thus the many-body localization is reached. If this fails, then self-consistently the system is able to act as its own bath and dephase local spin degrees of freedom by entangling them in high-order correlations spread across larger distances.

HOMEWORK

1. The scaling theory of localization states that

$$\beta(g) = \frac{d \ln g}{d \ln L} = d - 2 - \frac{a}{g} \tag{71}$$

where g is the dimensionless conductance, L is the system size, d is the dimensionality, and a is a known number of order 1. Let us assume $g = g_0$ for $L = l$. Calculate $g(L)$ for $L > l$, and $d = 1, 2, 3$.

[1] P. W. Anderson, Absence of Diffusion in Certain Random Lattices, Phys. Rev. 109, 1492 (1958).

[2] E. Abrahams, P. W. Anderson, D. C. Licciardello, and T. V. Ramakrishnan, Phys. Rev. Lett. 42, 673 (1979)