Notes on the superfluidity

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Keywords: superfluid; Bogovliubov transformation; single-mode approximation

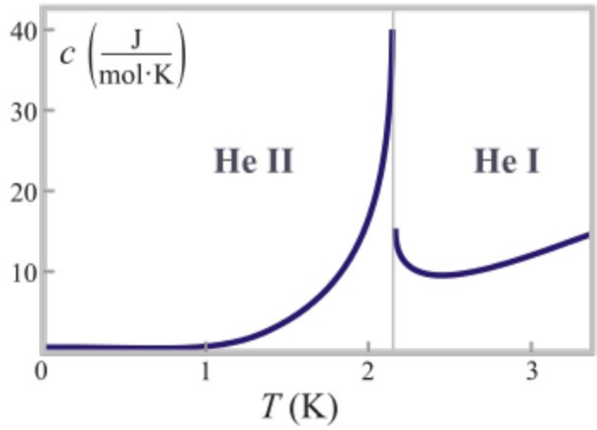

FIG. 1: The Lambda transition in liquid helium ^{4}He. The superfluid phase is known as He II.

In this chapter, we will consider superfluid in the interacting Bose systems. Superfluid shares the unusual property of zero dissipation. For example, we can imagine a toroidal pipe which we can fill with He-4. If we start He-4 supefluid flowing around the torus it will continue essentially forever. "Forever" here means an essentially unmeasurable long time.

In 1922-23 Onnes, along with some of his coworkers examined Helium. Looking at the heat capacity, they find something like Fig. 1. The diveregence in the heat capacity around 2.17K is a signal of a thermodynamic phase transition. The regular phase of helium is known as He I whereas the superfluid phase is known as He II.

The most remarkable thing about superfluid helium is the extremely good transport of heat in superfluids. In fact, going through the supefluid transition, thermal conductance can jump by a factor of 10^5 or more! In a superfluid, the thermal transport is so good that no region in the sample is at higher temperature than any other region. The system still has evaporation, but only directly from the surface.

In 1938 both Kaptiza and Allen (with Misener, another student from the Toronto lab) simultaneously discovered the effect that gives superfluid helium its name — that superfluid helium flows with apparently no resistance through very thin capillaries.

BOSE-EINSTEIN CONDENSATION

We start with the non-interacting bosons and review the concept of Bose-Einstein condensation (BEC).

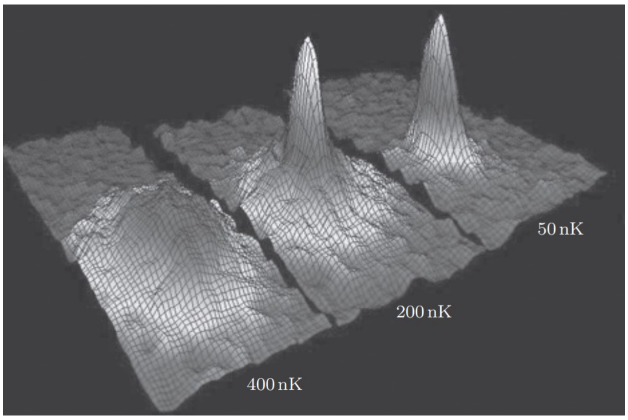

FIG. 2: Images of the velocity distributions of ultra-cold rubidium atoms. Left frame: temperature just above the Bose–Einstein condensation temperature; center frame, just below the condensation temperature with the appearance of a condensate; right frame, further lowering of temperature, resulting in almost all atoms being in the condensate.

We consider first non-interacting Boson:

$$H_0 = \sum_k (\frac{k^2}{2m} - \mu) b_k^\dagger b_k \tag{1}$$

where μ is chemical potential and $b_k^\dagger$ is bosonic operator.

Bosons satisfy the Bose-Einstein distribution

$$\frac{1}{e^{\beta\xi_k} - 1} = n_B(\xi_k) \tag{2}$$

Furthermore, the particle number is

$$N(\mu) = \sum_k n_B(\xi_k) = V \int \frac{d^3k}{(2\pi)^3} n_B(\xi_k) = V \int \frac{k^2 dk}{2\pi^2} n_B(\xi_k) = \frac{V(mT)^{3/2} Li_{\frac{3}{2}}(e^{\beta\mu})}{2\sqrt{2}\pi^{3/2}} \tag{3}$$

where the Jonquiere's function is

$$Li_{\frac{3}{2}}(b) = 4\pi b \int_0^\infty dx \frac{x^2}{e^{\pi x^2} - b} \tag{4}$$

To increase μ as lowering temperature T until $\mu = 0$, we have

$$N(\mu = 0) = N = Vc\frac{1}{\lambda_T^3}, \tag{5}$$

$$\lambda_T = \frac{1}{\sqrt{mT}}, c \approx 0.165. \tag{6}$$

Because we have set the chemical potential to zero, this is the largest possible density that can be achieved. Thus, if we can achieve a density greater than this, we will achieve BEC. For a given fixed physical density this means lowering the temperature below the critical temperature T_c, macroscopic number of particle occupy the ground state, and we call this phase as a Bose Einstein condensate. The critical temperature is estimated to be

$$N(0) = Vc\frac{1}{\lambda_{T_c}^3} = N \Rightarrow T_c = \frac{(cn)^{2/3}}{m}. \tag{7}$$

In 1995 BEC was directly observed in the momentum distribution of dilute gases made of alkali atoms, with a typical density of $10^{13} - 10^{15} cm^{-3}$. At such a low density the average distance between atoms is 1000A or larger, which is much bigger than the size of the atoms, and their interactions can thus be quite small. Such dilute gases are trapped using laser and/or magnetic fields, and cooled to astonishingly low temperatures (down to $100nK$ and in some cases even lower) using sophisticated evaporative cooling methods that take advantage of the specific optical and magnetic properties of the atoms.

GROSS-PITAEVSKII EQUATION

We consider Bose particles interacting through contact interaction

$$H = \int dr^3 \hat{\psi}^\dagger(\boldsymbol{r})[\frac{p^2}{2m} - \mu]\hat{\psi}(\boldsymbol{r}) + \frac{1}{2}\int dr^3 dr'^3 \hat{\psi}^\dagger(\boldsymbol{r})\hat{\psi}(\boldsymbol{r})U(\boldsymbol{r}-\boldsymbol{r}')\hat{\psi}^\dagger(\boldsymbol{r}')\hat{\psi}(\boldsymbol{r}') \tag{8}$$

From here we will further specialize to a particularly simple short-range interaction

$$U(\boldsymbol{r}-\boldsymbol{r}') = g\delta(\boldsymbol{r}-\boldsymbol{r}') \tag{9}$$

Such a short range interaction is actually a very good respresentation of the interaction for many physical bosonic systems. For example, for superfluid helium, the Helium atoms are strongly repulsive only at very short distance, and so this is actually a fairly good approximation. Similarly for modern cold-atom BECs, it is often the case that the delta function interaction is actually very representative of the physical system.

In terms of these field operators the plane wave creation operators are given by

$$a_k^\dagger = \frac{1}{\sqrt{V}}\int d\boldsymbol{r} e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\hat{\psi}^\dagger(\boldsymbol{r}) \tag{10}$$

The principle of a BEC is that we want to put many bosons in the same oribtal. Let us call this orbital ϕ, and write a creation operator ay ϕ that creates a boson in this orbital. Now let us write a coherent state for many bosons in this orbital.

$$|\alpha,\phi\rangle = e^{\alpha\hat{a}^\dagger_\phi}|0\rangle = (1+\alpha a^\dagger_\phi + \frac{\alpha^2(a^\dagger_\phi)^2}{2} + ...)|0\rangle = \sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle \tag{11}$$

This state is famous coherent state, which has the property

$$\hat{a}|\alpha\rangle = (0+\alpha|0\rangle + \frac{\alpha^2}{\sqrt{2!}}\sqrt{2}|1\rangle + \frac{\alpha^2}{\sqrt{3!}}\sqrt{3}|2\rangle ...) = \alpha|\alpha\rangle \tag{12}$$

where we used $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$. Thus, we can take a and replace it by α if the a operator is acting on the coherent state $|\alpha\rangle$. We then get the mapping $\hat{a}\to\alpha, \hat{a}^\dagger\to\alpha^*$. For example,

$$N \equiv \langle\hat{n}\rangle = \frac{\langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle}{\langle\alpha|\alpha\rangle} = \frac{|\alpha|^2\langle\alpha|\alpha\rangle}{\langle\alpha|\alpha\rangle} = |\alpha|^2 \tag{13}$$

which gives a physical meaning of $|\alpha| = \sqrt{N}$.

If applying the field operator, we have $\hat{\psi}(\boldsymbol{r})|\alpha\rangle = \phi(\boldsymbol{r})\alpha|\alpha\rangle$. We now calculate the expectation of the Hamiltonian in this coherent state

$$\begin{aligned}\langle H\rangle = \frac{\langle\alpha|H|\alpha\rangle}{\langle\alpha|\alpha\rangle} &= \int d\boldsymbol{r}[\phi^*(\mathbf{r})(-\frac{\nabla^2}{2m} + V(\boldsymbol{r}))\phi(\boldsymbol{r}) + \frac{g}{2}|\phi(\boldsymbol{r})|^4] \\ &= \int d\boldsymbol{r}[\frac{1}{2m}|\nabla\phi(\boldsymbol{r})|^2 + V(\boldsymbol{r})|\phi(\boldsymbol{r})|^2 + \frac{g}{2}|\phi(\boldsymbol{r})|^4]\end{aligned} \tag{14}$$

This expression is known as the Gross-Pitaevskii (or Ginzburg-Landau) form. This was first discussed by Vitali Ginzburg and Landau in 1950 in the context of superconductivity. In 1960 it was rederived by both Gross (in the west) and Pitaevskii (in the USSR) in the context of superfluid Helium.

Minimizing the energy by taking a functional derivative and setting it to zero $\delta H/\delta\phi^*(\boldsymbol{r}) = 0$, we obtain

$$[-\frac{\nabla^2}{2m} + V(\boldsymbol{r}) + \frac{g}{2}|\phi(\boldsymbol{r})|^2]\phi(\boldsymbol{r}) = 0 \tag{15}$$

which is known as the Gross-Pitaevskii equation, or non-linear-Schroedinger equation. This simply the single particle Schroedinger equation where there is an additional potential $g|\phi(r)|^2$ making the local potential higher in regions where there are many bosons. (What is the difference between Gross-Pitaevskii from the Schrodinger?)

THEORY OF WEAKLY-INTERACTING BOSE GAS

The approach we follow here, so-called Bogoliubov theory is a controlled approximation accurate in the limit of weak, but non-zero interaction.

We consider the Hamiltonian Eq. 8. Since the single particle eigenstaetes are plane waves we can conveniently write the field operator $\psi(\mathbf{r})$ in Fourier modes Eq. 10, we rewrite the Hamiltonian in the second quantization form

$$H = \sum_{\mathbf{k}} \frac{k^2}{2m} a_k^+ a_k + \frac{U}{2V} \sum_{k_{1,2,3,4}} a_{\mathbf{k}_1}^+ a_{\mathbf{k_2}}^+ a_{\mathbf{k}_3} a_{\mathbf{k}_4} \delta(\mathbf{k_1} + \mathbf{k_2} - \mathbf{k_3} - \mathbf{k_4}) \tag{16}$$

Now we expect that even when we turn on the interactions, the $k = 0$ mode should be macroscopically occupied at zero temperateure as it is in BEC. However the interaction must kick some of the particles out of the $k = 0$ state. Nonetheless, we still expect macroscopic occupancy of this orbital. Let us thus assume that the $k = 0$ orbital is in a coherent state with mean occupancy N_0 with $1 \ll N_0 \leq N$. At $T = 0$ if the interaction were zero, then $N_0 = N$. For sufficiently weak interaction, we might expect that $N - N_0 \ll N$.

Since we have a coherent state, as usual this allows us to replace operators with numbers via the usual substitution

$$a_0^+ \to \sqrt{N}_0, a_0 \to \sqrt{N}_0 \tag{17}$$

Returning to our Hamiltonian, we now have a small parameter $1/N_0$ and we can organize the parts of the Hamiltonian in terms of which has the most factors of N_0. The kinetic term has no factors of N_0 (since the kinetic energy of the $k = 0$ state is zero).

Next we consider the interaction term. Looking at the sum over k_1, k_2, k_3, k_4, the single term where $K_1 = k_2 = k_3 = k_4 = 0$ is the largest term, giving us four factors of $\sqrt{N_0}$ Thus the value of this term in the sum is simply the constant

$$\frac{U}{2V} N_0^2 \tag{18}$$

At next order we look for a term where there are three factors of $\sqrt{N_0}$ meaning three of the momentum are zero and one is nonzero. However, the momentum conservation condition in the interaction term forbids such situation. Due to the same reason, the terms with only one momentum non-zero all vanish, too.

At next order we look for a term where two of the momentum are zero and two are nonzero. Necessarily the momenta on the remaining two k's must appropriately sum to zero. We can choose the two non-zero k's to be both creation, both annihilation, or one of each (which can be chosen in four different ways). Thus the sum of all these terms can be written as

$$\frac{UN_0}{2V}\sum_{\mathbf{k}\neq 0}[4a^+_{\mathbf{k}}a_{\mathbf{k}} + a^+_{\mathbf{k}}a^+_{-\mathbf{k}} + a_{\mathbf{k}}a_{-\mathbf{k}}] \tag{19}$$

Finally we use one additional trick

$$N_0 = a^+_0 a_0 = N - \sum_{\mathbf{k}\neq 0} a^+_{\mathbf{k}}a_{\mathbf{k}} \tag{20}$$

plugged into the N_0, we obtain the resulting Hamiltonian

$$H = \frac{U\rho}{2}N + \sum_{\mathbf{k}\neq 0}[(\frac{k^2}{2m} + U\rho)a^+_{\mathbf{k}}a_{\mathbf{k}} + \frac{U\rho}{2}(a^+_{\mathbf{k}}a^+_{-\mathbf{k}} + a_{\mathbf{k}}a_{-\mathbf{k}})] \tag{21}$$

where the density is $\rho = N/V$. This Hamiltonian is quadratic (and therefore solvable), but it has so-called anomolous terms those with two creation or two annihilation operators. These terms allow particles to scatter in or out of the condensate (the state with $k = 0$). The scattering terms must conserve total momentum so you can only scatter two-in or two-out at a time.

Bogoliubov transformation

To solve the quadratic hamiltonian with anomolous terms, we invoke the so-called Bogoliubov transformation (invented 1947, by Bogoliubov). Let us write the following transformation

$$\begin{pmatrix} b_{\mathbf{k}} \\ b^+_{\mathbf{k}} \end{pmatrix} = \begin{pmatrix} \cosh\theta_k & \sinh\theta_k \\ \sinh\theta_k & \cosh\theta_k \end{pmatrix} \begin{pmatrix} a_{\mathbf{k}} \\ a^+_{-\mathbf{k}} \end{pmatrix} \tag{22}$$

It is easy to check that if the a's satisfy bosonic canonical commutations, then the b's similarly satisfy canonical commutations

$$[b_{\mathbf{q}}, b^+_{\mathbf{p}}] = \delta_{q,p}, \quad [b^+_{\mathbf{q}}, b^+_{\mathbf{p}}] = [b_{\mathbf{q}}, b_{\mathbf{p}}] = 0 \tag{23}$$

Making this transformation, the Hamiltonian becomes

$$\begin{aligned} H = const + \frac{1}{2}\sum_{\mathbf{k}\neq 0}&[(\frac{k^2}{2m} + U\rho)\cosh(2\theta_k) - U\rho\sinh(2\theta_k)](b^+_{\mathbf{k}}b_{\mathbf{k}} + b^+_{-\mathbf{k}}b_{-\mathbf{k}}) \\ &- [(\frac{k^2}{2m} + U\rho)\sinh(2\theta_k) - U\rho\cosh(2\theta_k)](b^+_{\mathbf{k}}b^+_{-\mathbf{k}} + b_{-\mathbf{k}}b_{\mathbf{k}}) \end{aligned} \tag{24}$$

If we choose

$$(\frac{k^2}{2m} + U\rho)\sinh(2\theta_k) - U\rho\cosh(2\theta_k) = 0 \tag{25}$$

the anomolous terms are eliminated and we can diagonalize the Hamiltonian

$$H = const + \sum_{\mathbf{k}\neq 0} E_{\mathbf{k}} b^+_{\mathbf{k}} b_{\mathbf{k}} \tag{26}$$

where

$$E_{\mathbf{k}} = \sqrt{(\frac{k^2}{2m} + U\rho)^2 - (U\rho)^2} \approx \sqrt{\frac{U\rho}{m}}|k| + ..., (\mathbf{k} \to 0) \tag{27}$$

Note that this spectrum of excitations in linear in k at low k and then curves to be quadratic at large k. Because the dispersion is linear it satisfies the Landau criterion for superfluidity! (see below)

The excitations created by the by b^+_k operators are sometimes known as bogoliubons. The ground state is obviously given by the state with no bogololiubons present

$$b_{\mathbf{k}}|0\rangle = 0, \forall k \neq 0 \tag{28}$$

Note that b_k is hybridization of a_k and a^+_k, meaning that in the ground state there is some occupation of bosons in $k \neq 0$ orbitals. As we predicted previously, the interaction has pushed some of the bosons out of the $k = 0$ orbital.

LANDAU'S PICTURE FOR SUPERFLUIDITY

What is the origin of fiction?

The current flow of superfluid is dissipationless, i.e. the property of flowing through narrow capillaries or slits without exhibiting viscosity. How to understand it? Generally speaking, no energy can be exchanged if there is no elementary excitation to create. In reality, this means that the excitations of the system are energetically so high-lying that the kinetic energy stored in the current-carrying particles is insufficient to create them. But this is not the situation that we encounter in the superfluid (as shown in the above section, there is low-energy Goldstone mode in superfluidty). As we saw above, there is no energy gap separating the quasi-particle excitations of the system from the ground state. Rather, the dispersion vanishes linearly as $k \to 0$.

Next we present the idea from Landau. Let us assume further that the liquid is flowing along a capillary at a constant velocity v_0. In a standard viscous liquid we expect that due to the friction against the walls of the tube and the friction within the liquid itself, the presence of viscosity would lead to dissipation of the kinetic energy and the flow would gradually become slower.

Let us suppose that a single elementary excitation appears in the liquid, with momentum $\mathbf{p}$ and energy ϵ_p. In the liquid coordinate system,

$$E^{liquid} = \epsilon_p, \quad \mathbf{P}^{liquid} = \mathbf{p} \tag{29}$$

Return to the lab coordinate system, we use the Galilean transformation back to the laboratory frame

$$E^{lab} = \frac{1}{2}m(\mathbf{v}+\mathbf{v}_0)^2 = E^{liquid} + \mathbf{P}^{liquid}\cdot\mathbf{v}_0 + \frac{1}{2}mv_0^2, \quad \mathbf{P}^{lab} = \mathbf{P}^{liquid} + m\mathbf{v}_0 \tag{30}$$

Next our question is whether in the lab frame, the energy to create an excitation is ever negative: $\epsilon_p + \mathbf{P}^{liquid}\cdot\mathbf{v}_0 < 0$. (this is the change in energy due to the appearance of the excitation). If so, excitaions are created spontaniously and energy is dissipated.

If its minimum is negative, $\epsilon_p - pv_0 < 0$, i.e. $v_c < v_0$, the dissipation occurs. Inversely, the system transports charge without dissipation for velocities smaller than the critical velocity

$$v_0 < v_c, \quad v_c = \min\{\frac{\epsilon_p}{p}\}. \tag{31}$$

Below this critical velocity, there is no way to create a quasiparticle while conserving energy and momentum (see Fig. 4).

This simple argument by Landau is of fundamental importance for the understanding of a superfluid. A direct consequence is that systems where $\min\{\frac{\epsilon_p}{p}\} = 0$ cannot be superfluid since then $v_c = 0$ and an arbitrarily small velocity would result in dissipation. We can write the minimum of $\frac{\epsilon_p}{p}$ as the solution of

$$0 = \partial_p(\frac{\epsilon_p}{p}) \Rightarrow \epsilon_p = vp \tag{32}$$

For a given point on the curve ϵ_p we are thus asking whether the slope of the curve is identical to the slope of a straight line from the origin through the given point. The slope of the line at the touching point is the critical velocity according to Landau above which superfluidity is destroyed. In particular, any gapless dispersion with slope zero in the origin must lead to dissipation for any nonzero velocity. The criterion rather requires a nonzero critical velocity v_c and the existence of a condensate. Without a condensate, there is nothing to transport the charge without friction. We shall see later that Bose-Einstein condensation is always accompanied by a gapless mode $E_p = 0$ due to the Goldstone theorem, and this gapless mode is called Goldstone mode. if for instance the dispersion of the Goldstone mode is linear, $E_p \sim p$, the mode is gapless but Landau's critical velocity is nonzero, and in fact identical to the slope of the Goldstone mode. Typically, the slope of a Goldstone mode is indeed linear for small momenta. This is true for instance in superfluid helium. On the other hand, if we had $E_p \sim p^2$ for small momenta, the slope of the dispersion at the origin would be zero and as a consequence $v_c = 0$.

Free boson is not superfluid

A free Boson condensate with quadratic dispersion is not a superfluid. Note that, for any velocity v_0, we can always have small enough $k \neq 0$ such that $k^2/2m + \frac{1}{2m}\mathbf{k}\cdot\mathbf{v}_0 < 0$. That is, a moving free boson liquid can always lower its energy by creating excitations. The velocity of the liquid is gradually reduced in the process. Therefore, a free boson condensate is not a superfluid.

In contrast, a Boson fluid with phonon-like excitation spectrum is a superfluid. Therefore, superfluidity is a result of Bose condensation (a large number of atoms remain in the same state) + repulsive interaction between bosons (phonon-like excitation spectrum). It would not exist without the two factors working together.

Various consequences

The conserved current is

$$J = \frac{\rho_0}{m}\nabla\phi. \tag{33}$$

We integrate over a closed path,

$$\oint \nabla\phi = \phi(2\pi) - \phi(0) = 2\pi n \tag{34}$$

$$\Rightarrow \int v dl = \int \frac{J}{\rho_0} dl = hn/m \tag{35}$$

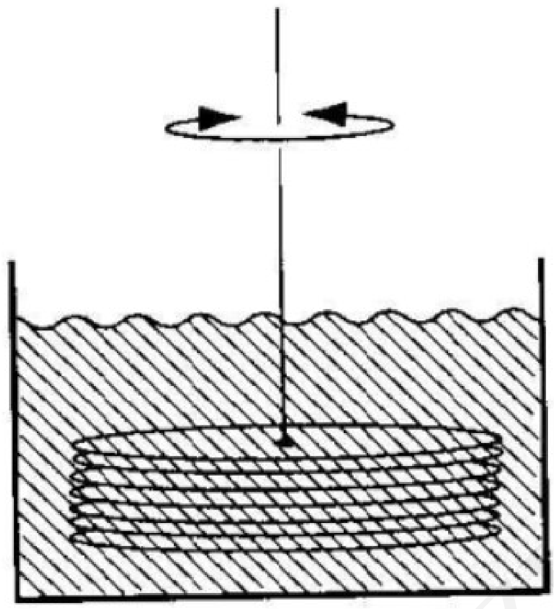

FIG. 3: The Andronikoshvili experiment. Since normal fluid sticks between the closely spaced disks (but superfluid does not), the moment of inertia of the stack of disks, and hence frequency of oscillation of the disks tells us the fraction of fluid that is superfluid.

where the superfluid velocity is quantized.

If the vortex is allowed, this current can be further expressed by

$$J = \rho_s \nabla \phi = \rho_s \frac{n}{r} \hat{\theta} \tag{36}$$

In this case we can have the phase θ wrap by any integer multiple of 2π as we go around this single point. This is known as a "vortex". The existence of vortices in superfluid helium was predicted by Feynmann in 1955.

SUPERFLUID DENSITY AND THE ANDRONIKOSHVILI EXPERIMENT

Landau made a prediction for the superfluid density then convinced a young experimentalist, Elephter Andronikoshvilli, to make careful measurements of the superfluid density in 1946, and the results supported his predictions.

The experiment is shown in Fig. . A stack of closely spaced disks is hung in a container of helium by a thin wire which acts as a torsion oscillator. The idea is that when the stack of disks rotates, normal fluid, which is viscous will get stuck between the disks and must rotate with the stack. However superfluid, which has no viscosity slips through the closely spaced disks and does not rotate. The normal fluid thus contributes to the total moment of inertia of the stack, and hence changes the oscillation frequency of the torsion oscillator. By measuring the change in the oscillation frequency as a function of temperature, one can determine the fraction of helium that is superfluid as a function of temperature.

Andronikoshvili clearly measured that the normal fluid density is proportional to $\rho_N \sim T^4$ at low temperature.

In Landau's picture, again this calculation relies on thinking about superfluids in both the lab and the moving superfluid frame. Here, he realized that the normal fluid is dragged by the wall and will have the same velocity as the wall, whereas the superfluid moves seperately.

As before, in the lab frame we have

$$\epsilon^{lab-frame} = \epsilon_p - vp \tag{37}$$

where ϵ_p describes the excitation spectrum in the superfluid frame. As mentioned in the above section where we calculated critical velocity, if $\epsilon_{lab-frame}$ becomes negative, then we get spontaneous generation of excitations and we get dissipation. However, even if $\epsilon_{lab-frame} > 0$, the excitations can still be excited thermally. We expect that the density of such particles will be given by $n_B(\epsilon_p - vp)$.

Next we calculate the current carried by these excitations:

$$j = \int \frac{d^3p}{(2\pi\hbar)^3}\mathbf{p}n_B(\epsilon_p - \mathbf{v}\cdot\mathbf{p}) \tag{38}$$

$$\to \int \frac{d^3p}{(2\pi\hbar)^3}\mathbf{p}n_B(\epsilon_p) - \mathbf{p}\beta(\mathbf{v}\cdot\mathbf{p})n'_B(\beta\epsilon_p) \tag{39}$$

In the second line we take the small v limit. The first term vanishes by the symmetry. Next we estimate the second term.

Non-interacting Boses

Let us try plugging in the dispersion $\epsilon_p = p^2/2m$ for a noninteracting BEC. We can obtain

$$\begin{aligned} \rho_N = J_x/v_x &\sim \beta \int \frac{d^3p}{(2\pi\hbar)^3}p_x^2 n'_B(\beta p^2/2m) \\ &= \beta\beta^{-5/2}\int\int \frac{d^3q}{(2\pi\hbar)^3}q_x^2 n'_B(q^2/2m) \sim T^{3/2} \end{aligned} \tag{40}$$

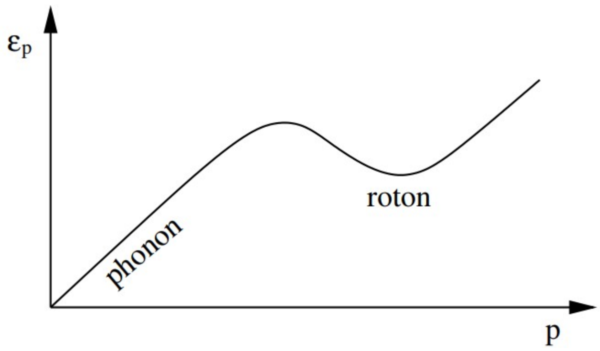

FIG. 4: Schematic plot of the Goldstone dispersion for superfluid helium. This mode is often modelled in terms of two different modes, the phonon and the roton.

Interacting Bosons

For an interacting BEC or superfluid, we expect a low energy acoustic mode $\epsilon = vp$.

$$\begin{aligned}\rho_N = J_x/v_x &\sim \beta \int \frac{d^3p}{(2\pi\hbar)^3} p_x^2 n_B'(\beta vp) \\ &= \beta\beta^{-5} \int\int \frac{d^3q}{(2\pi\hbar)^3} q_x^2 n_B'(vq) \sim T^4\end{aligned} \tag{41}$$

in agreement with the results of Andronikoshvili.

ROTOR MINIMAL FROM FEYNMANN

We further study the low-energy excitation of liquid helium-4.

Suppose the ground state wave function is $|\Phi_0\rangle$, Feynmann proposed a trial excited state as

$$|\Psi_k\rangle = \frac{1}{\sqrt{N}} \rho_k |\Phi_0\rangle \tag{42}$$

where ρ_k is the density operator at wave vector $\vec{k}$:

$$\rho_k = \sum_i e^{i\mathbf{k}\cdot\mathbf{r}_i} \tag{43}$$

This is just the Fourier transformation of density operator $\rho(\mathbf{r}) = \sum_i \delta(\mathbf{r} - \mathbf{r}_i)$.

It is surprising, this trial wave function works so well in liquid helium-4. What we would like to calcualte is the energy of the excitation compared to the ground state which we write

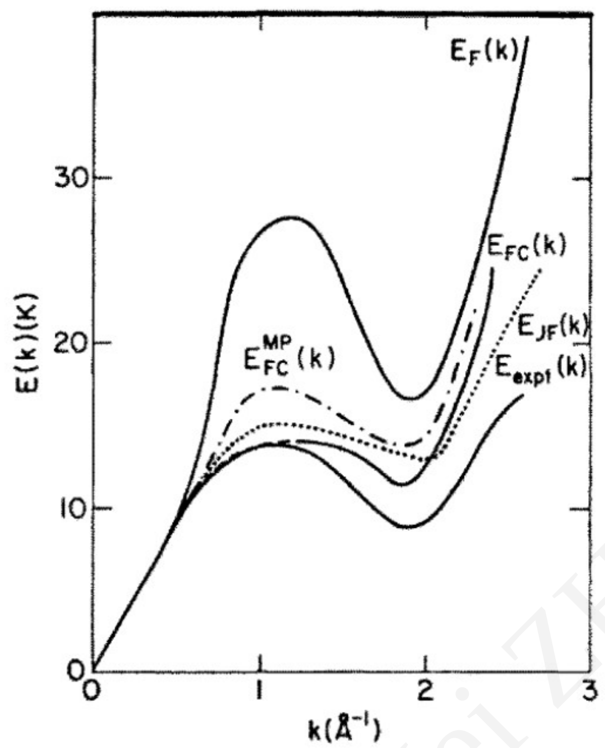

FIG. 5: Feynman Theory and Low energy excitations of Helium. Top curve is prediction of Feynman theory. Bottom curve is experimental measurement of the excitation spectrum.

as

$$\Delta_k = E_k - E_0 = \frac{\langle \Psi_k | H - E_0 | \Psi_k \rangle}{\langle \Psi_k | \Psi_k \rangle} \equiv \frac{f(k)}{S(k)} \tag{44}$$

The denominator is actually the structure factor:

$$\begin{aligned} S(k) &= \frac{1}{N} \langle \Phi_0 | \rho_k^\dagger \rho_k | \Phi_0 \rangle \\ &= \frac{1}{N} \langle \Phi_0 | \sum_{ij} e^{-ik(r_i - r_j)} | \Phi_0 \rangle = FT[\langle \Phi_0 | \sum_{ij} \delta(r - (r_i - r_j)) | \Phi_0 \rangle] \end{aligned} \tag{45}$$

Experimental neutron diffraction measurements of the structure factor of Helium-4 shows, the peak in the structure factor is roughly the analog of a Bragg peak for a crystal. It occurs at a wavevector of roughly $2\pi/a$ where a is the typical inter-particle spacing.

Next we turn to the numerator,

$$\begin{aligned} f(k) &= \frac{1}{N} \langle \Phi_0 | \rho_k^\dagger (H - E_0) \rho_k | \Phi_0 \rangle \\ &= \frac{1}{N} \langle \Phi_0 | \rho_k^\dagger H \rho_k | \Phi_0 \rangle - \langle \Phi_0 | \rho_k^\dagger \rho_k H | \Phi_0 \rangle = \frac{1}{N} \langle \Phi_0 | \rho_k^\dagger [H, \rho_k] | \Phi_0 \rangle \end{aligned} \tag{46}$$

On the other hand, we equivalently could have written

$$f(k) = -\frac{1}{N}\langle\Phi_0|[H,\rho_k]\rho_k^\dagger|\Phi_0\rangle \tag{47}$$

Putting them together we obtain the double commutator form

$$f(k) = \frac{1}{2N}\langle\Phi_0|[\rho_k^\dagger,[H,\rho_k]]|\Phi_0\rangle \tag{48}$$

We now must determine the double commutator, which will turn out to be a simple number rather than an operator! To do this we notice recall the Hamiltonian has three terms, a kinetic term K, a one body potential term V and a two body interaction term U (it would not matter if we had three or four body terms etc). We write $H = K + V + U$, where $K = \sum_i \frac{-\nabla_i^2}{2m}$, $V = \sum_i V(r_i)$, $U = \sum_{ij} U(r_i - r_j)$. Here density operator ρ only contains the operator r and this then commutes with both U and V which also only contain r. Thus, ρ_k does not commute with the kinetic term, so we simplify

$$[H,\rho_k] = [K,\rho_k] = [\sum_i \frac{-\nabla_i^2}{2m}, \sum_j e^{ikr_j}] = -\frac{1}{2m}\sum_j(-k^2 + 2ik\nabla_j) \tag{49}$$

and

$$\begin{aligned}
[\rho_k^\dagger,[H,\rho_k]] &= -\frac{1}{2m}\sum_j[\rho_k^\dagger,(-k^2+2ik\nabla_j)] \\
&= -\frac{1}{m}\sum_j[\rho_k^\dagger, ik\nabla_j] = -\frac{N}{m}[e^{-ikr}, ik\nabla] \\
&= \frac{k^2 N}{m}
\end{aligned} \tag{50}$$

which is simply a scalar rather than an operator. Thus we obtain

$$f(k) = \frac{k^2}{2m} \tag{51}$$

Now plugging back into our original formula for the excitation energy

$$\Delta_k = \frac{f(k)}{S(k)} = \frac{k^2}{2mS(k)} \tag{52}$$

This is a rather remarkable result! In this approxiation, the low energy excitation spectrum is completely determined by the structure factor.

In Fig. 5 the top curve is the prediction of the Feynman theory with the experimentally measured structure factor $S(k)$ as input. The bottom curve is the experimentally measured excitation spectrum (using inelastic neutron scattering). The structure of the two curves is quite similar.

THERMODYNAMICS OF SUPERFLUID HELIUM

While the existence of a Goldstone mode and the linearity at small p are very general features, the details of the complete dispersion of this mode depend on the details of the interactions in a given system. In superfluid helium, it turns out that the mode has a dispersion of the form shown in Fig. For low energies, it can effectively be described by two different excitations, one accounting for the linear low-momentum part – this is called the "phonon" – and one accounting for the vicinity of the local minimum at a finite value of p – this is called the "roton". We can write these two dispersions as

$$E_p = cp, (\text{“}phonon''') \tag{53}$$

$$E_p = \Delta + \frac{(p-p_0)^2}{2m}, (\text{“}roton''') \tag{54}$$

with parameters c, D, p_0, m, whose values are specified. Let us first compute some of the thermodynamic properties given by the Goldstone mode. We start from the general expression for the pressure,

$$P = -T\int \frac{d^3p}{(2\pi)^3}\ln(1-e^{\epsilon_p/T}) = \frac{1}{3}\int \frac{d^3p}{(2\pi)^3}p\frac{\partial \epsilon_p}{\partial p}n_B(\epsilon_p) \tag{55}$$

Consequently, the phonon contribution to the pressure is

$$P_{ph} = \frac{v}{6\pi^2}\int_0^\infty dp\frac{p^3}{e^{vp/T}-1} = \frac{T^4}{6\pi^2 v^3}\int_0^\infty dy\frac{y^3}{e^y-1} = \frac{\pi^2 T^4}{90v^3} \tag{56}$$

And we can calculate the specific heat as

$$c_{V,ph} = T\frac{\partial s_{ph}}{\partial T} = \frac{2\pi^2 T^3}{15v^3} \tag{57}$$

The calculation of the roton contribution is a bit more complicated,

$$P_{rot} = \frac{1}{6\pi^2 m}\int_0^\infty dpp^3\frac{p-p_0}{e^{\epsilon_p/T}-1} \tag{58}$$

In the limit of $T \ll \Delta$, we have a form as

$$P_{rot} \approx \frac{e^{-\Delta/T}}{6\pi^2 m}\int_0^\infty dpp^3(p-p_0)e^{-\frac{(p-p_0)^2}{2mT}} \approx \sqrt{\frac{m}{2\pi^3}}p_0^2 T^{3/2}e^{-\Delta/T} \tag{59}$$

Homework: Please prove that the bogoliubov transformation (Eq. 23) doesnot change the statistics of particles.