Toric code model: Z_2 topological order

W. Zhu[1]

[1] *Westlake Institute of Advanced Study,*

*Westlake University, Hangzhou, P. R. China**

Contents

[2]Reporting of typos, inaccuracies and errors to zhuwei@westlake.edu.cn would be greatly appreciated.

Having outlined the framework for describing condensed matter phases that can be captured within the paradigm of spontaneously symmetry breaking, we now proceed to discuss the basic features of quantum spin liquids which are examples of a quantum ordered phase. Instead of proceeding with a general discussion, we shall focus on the following central features of such quantum spin liquids:

1. Topological order as reflected in presence of long range quantum entanglement of the many-body ground state.

2. Topological ground state degeneracy and non-trivial statistics of the excitations.

3. Fractionalization of microscopic quantum numbers.

4. Emergent gauge fields, like a photon-like gapless excitation, whose gaplessness is robust to symmetry breaking perturbations.

As with so many great ideas in this field, the Toric code was invented by Kitaev (Kitaev 1997). The toric code is a canonical example of a system with topological order and anyonic excitations. It also establishes a connection between physical systems and quantum information, which motivates the study of anyons and topological order as a route to fault-tolerant quantum computation.

MODEL

Consider a model of spin-1/2s sitting on the bonds of a square lattice, as shown in Fig. 1. The interacting Hamiltonian reads

$$H = -\sum_s V_s - \sum_a P_a \tag{1}$$

where s and a denote the sites and plaquette of the square lattice respectively, as shown in Fig. 1, and

$$V_s = \prod_{i \in s} \sigma_i^z, \qquad P_a = \prod_{i \in a} \sigma_i^x \tag{2}$$

Note that this Hamiltonian does not have spin rotation or lattice symmetries in general when the coupling constants differ from site/plaquette to site/plaquette. All symmetries like time reversal that is left can also be broken by adding small perturbations and still the following discussions will hold.

This model was introduced by Kitaev in 1997. The "toric" means torus, i.e. this model is meaningful for quantum computation on the torus geometry. The "code" means that this model has application in the quantum computation.

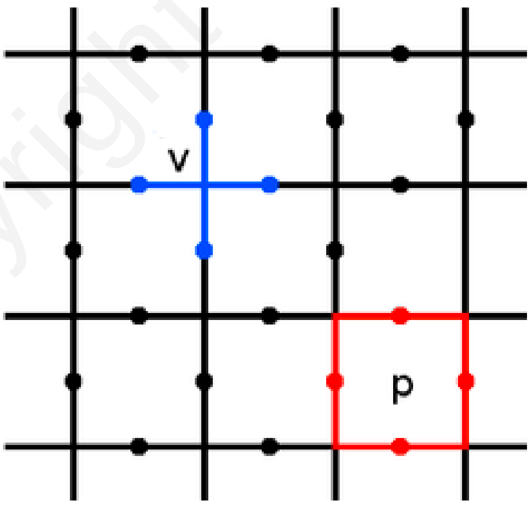

FIG. 1: The Toric code model on a square lattice. The spin-1/2s live on the bonds (denoted by filled circles). The blue (red) simplex around a site (plaquette) denotes operator V_s (P_a) for the site (plaquette) as defined in the main body of the text.

Vertex and Plaquette Operators

The vertex operator simply counts the parity of the number of down spins (number of colored edges) incident on the vertex. It returns 1 if there are an even number of incident down spins at that vertex and returns -1 if there are an odd number. (And in either case, as is obvious $V^2 = 1$). There are a total of $N = N_x N_y$ vertex operators.

We now define a slightly more complicated operator known as the plaquette operator. which involves the spins on all of the edges of the plaquette as depicted in Fig. 1. There are a total of $N_x N_y$ plaquette operators. In the basis we are using, the spin-up/spin-down basis (along z-direction), the P_a operator is off-diagonal, it flips spins around a plaquette.

All of the plaquette operators and all of the vertex operators commute with each other. It is obvious that

$$[V_s, V_{s'}] = 0 \tag{3}$$

since V_s are only made of z operators and all of these commute with each other. Similarly,

$$[P_a, P_b] = 0 \tag{4}$$

since P_a are made only of σ^x operators and all of these commute with each other. The nontrivial statement is that

$$[V_s, P_a] = 0 \tag{5}$$

for all s and a. The obvious case is when V and P do not share any edges, then the two operators obviously commute. When they do share edges, geometrically they must share exactly two edges, in which case the commutation between each shared σ_i^x and σ_i^z accumulates a minus sign, and there are exactly two shared edges so that the net sign accumulated is 1 meaning that the two operators commute.

Hilbert space

We imagine an N_x by N_y square lattice with spins on each edge, where the edges of the lattice are made periodic hence forming a torus (periodic boundary conditions). The number of squares is $N_x N_y$. The total number of spins is $N = 2N_x N_y$ and correspondingly the dimension of the Hilbert space is $2^N = 2^{2N_xN_y}$. We will work with a basis in our Hilbert space of up and down spin along z-direction.

We can start by noting that V_s and P_a are Hermitian and square to the identity, therefore have eigenvalues ± 1. For the ground state, we target the conditions

$$V_s = 1, P_a = 1, \forall s, a \tag{6}$$

which are $N_xN_y + N_xN_y = 2N_xN_y$ conditions in total. Now, we take $V_s = 1$ and $P_a = 1$ as constraints on this Hilbert space, i.e. violating any constraint costs energy, so the ground state should obey every constraint. But, these conditions are not independent, because of (all spins are used by twice)

$$\prod_s V_s = 1, \prod_a P_a = 1 \tag{7}$$

So there are only $2N_xN_y - 2$ independent conditions in total.

Putting everything together, we have $2N_xN_y - 2$ constraints, so we expect a ground state manifold of dimension $2^{2N_xN_y-(2N_xN_y-2)} = 4$, which is the topological degeneracy of the ground state. Please note that, this degeneracy is not from the symmetry broken. Its origin is topology.

Ground state

The ground state should be every vertex with $V_s = 1$. So each vertex has even spin-*up*. Clearly the state with all edges $|\uparrow\rangle$ has eigenvalue $+1$ for all V_s. If a single spin is flipped, the resulting state will have eigen-value -1 for the V_s immediately adjacent to that edge. In fact, anytime a vertex is surrounded by an odd number of $|\downarrow\rangle$, the corresponding V_s will have eigenvalue -1. The only states which do not have at least one such vertex are those where the number of $|\downarrow\rangle$ is even and the $|\downarrow\rangle$ form closed loops (think about down-edge as a string), so the set of all such states forms the $+1$ eigenspace of the V_s operators.

Clearly there are many, many possible states with closed loops, so the ground state is massively degenerate if we only satisfy the V_s constraints. To fix this degeneracy we add the P_a terms to the picture. Each P_a is made up of σ^x operators, so it will flip all the spins on the border of a given plaquette. This has the effect of adding a new loop of occupied edges or smoothly deforming an existing loop, in the sense that we cannot break or join a string of occupied edges by applying any combination of P_a.

So, the ground state can be expressed as an equal superposition of states composed of all closed loops of down spins living in a sea of up spins:

$$|\psi_{GS}\rangle = \sum_{loops} |loops\rangle = \prod_{a} \frac{1+P_a}{2} |\{\uparrow_i\}\rangle \tag{8}$$

where each state $|loops\rangle$ is made by applying P_a on the state with all spin-ups: $|\uparrow_1\uparrow_2 ...\rangle$.

Let us consider in some detail what operator P_a does when it hits a particular loop configuration (or state). If none of the four bonds forming the square $a = p$ is involved in any existing loops then P_p creates a small loop around p. For similar reasons, if such a small loop already exists, then P_a annihilates it. Most interestingly, if a subset of these four bonds consists of parts of other loop(s), then P_a changes their configurations (by inserting a "detour" into the path). We thus find that the operation $\sum_p P_p$ will generate a very complicated superposition of a huge number of loop configurations with arbitrarily large loops, even if the initial configuration in $|loop\rangle$ contains no loop at all (which is the case if we start with the state in which all spins are up. Such a massive superposition creates a long-range entangled and topologically ordered state, with properties identical to those of the Z2 RVB spin liquid state.

Next, how many different (inequivalent) ground states on a torus? A little thought should convince you that a string of occupied edges that winds all the way around the

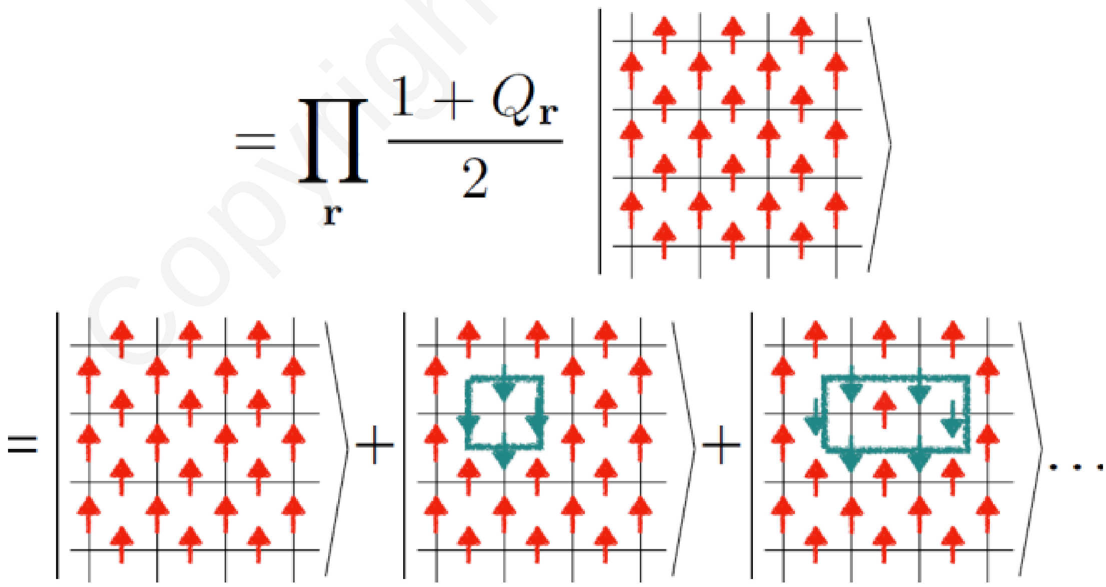

FIG. 2: The configurations in the ground state.

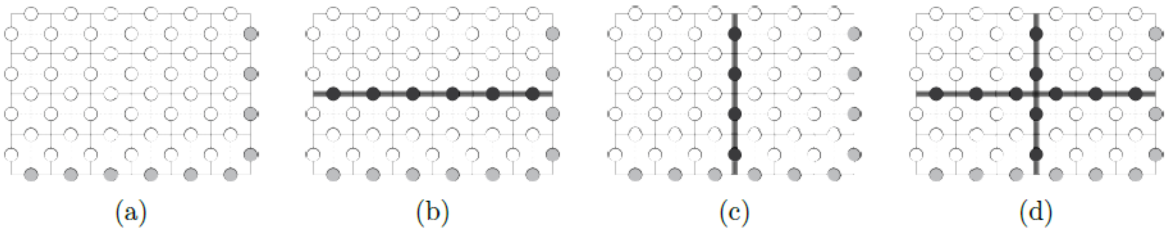

FIG. 3: Examples of the four classes of loops which cannot be deformed into each other. This also defines the Wilson-loop operators which defines four-fold ground state manifold.

torus in one direction cannot be contracted into nothing, as this would require breaking the string somewhere. This realization gives us 4 classes of loops which cannot be deformed into one another: the trivial class with no loops, a loop that winds around the torus in either direction, and the combination of two loops with one winding around in each direction (Fig. 3).

The fact that there are 4 inequivalent loop configurations should not be too surprising; as we mentioned, they correspond to the 4-dimensional GS manifold. Note that this 4-fold degeneracy is directly related to the topology of the system (i.e. the periodic boundary conditions); we say that the toric code displays topological order, which is in contrast to the degeneracy from the symmetry reason.

In order to destory this ground state manifold, say change from one ground state to another, one needs to flip L spins along the closed loop, which relates to the energy cost 2^L. This energy cost is too high, so the ground state manifold is quite robust.

Excitations

Now that we know the ground state, we can try to characterize the low-energy excitations. The most elementary excitation is thus a lattice site with $V_s = -1$, or a square with $P_a = -1$. These are the two types of quasiparticles of the system. The first lives on a lattice site, and we will call this excitation "charge", while the second lives in a plaquette (which is also the smallest closed loop), and we will call it "flux", in obvious analogy with electromagnetism (see Fig. 4). The big difference here is that the charge and flux can only take values 0 or 1, and form (separate) Z2 sets under addition. (That is charge and flux are only conserved modulo 2.)

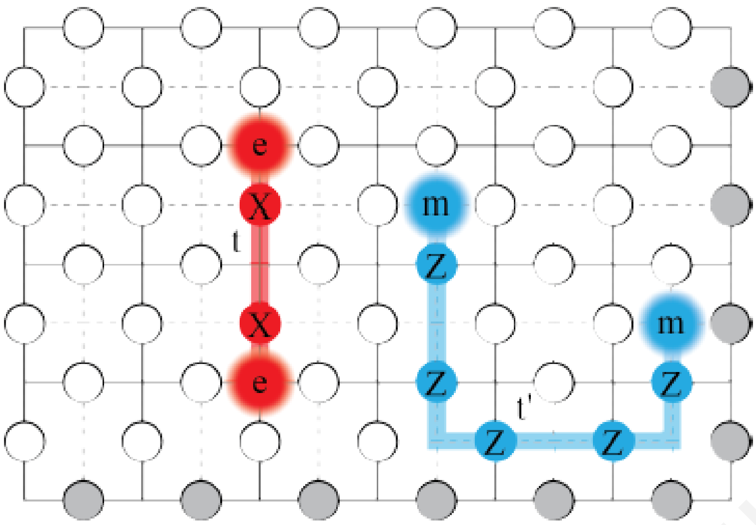

FIG. 4: Strings of X or Z operators applied to the ground state create pairs of "electric" or "magnetic" excitations e and m. The energy of each pair is +2 because the end of any string anticommutes with one V_s or P_a.

At this point it may seem that these elementary charge or flux excitations are very similar to a single (local) spin flip in a ferromagnet. Again this is not the case. In fact a single excitation cannot be created by any local operations. Let us consider a charge excitation first. To create one at site s, it appears that all one need do is flip the spin on one of the bonds connected to s using σ_x. It is easy to convince oneself that this operation actually creates a pair of charges on neighboring sites s and s', connected by the bond (see Fig. 4). Additional operations may separate these two charges, but never eliminate one of them (see Fig. 4). We thus find that charges can only be created in pairs. Exactly the same is true for flux excitations. Of course, mathematically this property is guaranteed by the constraints (Fig. 4).

Physically, the somewhat strange properties discussed above reflect the fact that these are topological excitations (or defects in the topological order) that are intrinsically non-local. Such non-locality is reflected by the fact that these excitations can feel the existence of each other even when they are very far apart. Naively there is no interaction among them, as the energy of the system is independent of their separations. However, as we demonstrate below, if one drags a charge around a flux, the system picks up a non-trivial Berry phase very much like that in the Aharonov–Bohm effect (hence the names charge and flux!). There is thus a "statistical interaction" between a charge and a flux, which is of topological nature.

The easiest way to do this is to think in terms of the operators that create an excitation. These turn out to be string operators:

$$S^x(t) = \prod_{j\in t} \sigma_j^x, \quad S^z(t') = \prod_{j\in t'} \sigma_j^z \tag{9}$$

where $t(t')$ is a string of edges on the (dual) lattice (Fig. 4). If a string t is open-ended, the σ^x operators at its endpoints will anticommute with one V_s each, raising the energy above the ground state by 2. Similarly, the ends of a dual string t' anticommute with P_a. Note that it is impossible to create an excitation with unit energy. More intuitively, it is impossible to create an open string with one end, so the only allowed excitations come in pairs. We call the quasiparticle pairs created by $S^x(t)$ "electric charges" e and the pairs created by $S^z(t')$ "magnetic vortices" to match existing literature on gauge field models, but this is just a convention.

The most interesting thing about these quasiparticles is their exchange statistics. Clearly $S^x(t_1)$ commutes with $S^x(t_2)$ for any strings t_1 and t_2, as they only involve σ^x terms. The same is true of $S^z(t)$, so it appears that both the e and m particles are hard-core bosons. Next we consider the loop string, i.e. a loop $\mathcal{L}$ that starts and ends at s. We can drag the charge around this loop using the operator $\prod_{j\in\mathcal{L}} \sigma_j^x$ (where $j \in \mathcal{L}$ means that the bond j is part of the loop $j \in \mathcal{L}$), resulting in the final state

$$|\text{finial}\rangle = \prod_{j\in\mathcal{L}} \sigma_j^x |\text{initial}\rangle \tag{10}$$

Using the fact that $\prod_{j\in\mathcal{L}} \sigma_j^x = \prod_{p\in A} V_p$, where A is the region enclosed by $\mathcal{L}$, we find

$$|\text{finial}\rangle = |\text{initial}\rangle \tag{11}$$

However, if we move an e particle around an m particle or vice versa (Fig. 5), i.e. when a flux excitation inside the loop, it acquires a phase of -1 from the overlapping string operators,

$$|\text{finial}\rangle = -|\text{initial}\rangle \tag{12}$$

meaning that the charge picks up a -1 Berry phase when circling around the flux (or vice versa). Quasiparticles that behave in this way are called anyons, because they fall outside the usual bosonic/fermionic picture.

Anyonic statistics and topological order

The braiding and statistics of quasiparticles are encoded in the $\mathcal{S}$ and $\mathcal{U}$ matrices [X. G. Wen 1990]. For Abelian phases, the element $\mathcal{S}_{ij}$ corresponds to the exchange phase when the i'th quasiparticle encircles the j'th quasiparticle. The $\mathcal{U}$ matrix is diagonal and the element $\mathcal{U}_{ii}$ corresponds to the exchange phase of the i'th quasiparticle with an identical one. In the topological quantum field theory, the modular $\mathcal{S}$ and $\mathcal{U}$ matrices describe the action of certain modular transformations on the degenerate ground states.

Taking into account the conditions that the $\mathcal{S}$ matrix is symmetrical ($\mathcal{S}_{ij} = \mathcal{S}_{ji}$) and $\mathcal{U}$ is a diagonal matrix, one obtains

$$\mathcal{S} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \quad \mathcal{U} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

which is from the conformal field theory [Verlinde 1986, Zhenhan Wang 2004]. Through the modular matrices above, we have the full information of Z_2 topological order:

1. There are four types of quasiparticles, which are denoted as 1 (identity), s, v and sv.

2. The first row and column of $\mathcal{S}$ correspond to quasiparticle quantum dimension as $\mathcal{S}_{i1} = d_i/\mathcal{D}$, where d_i quasiparticle's individual quantum dimension and $\mathcal{D} = \sqrt{\sum_i d_i^2}$

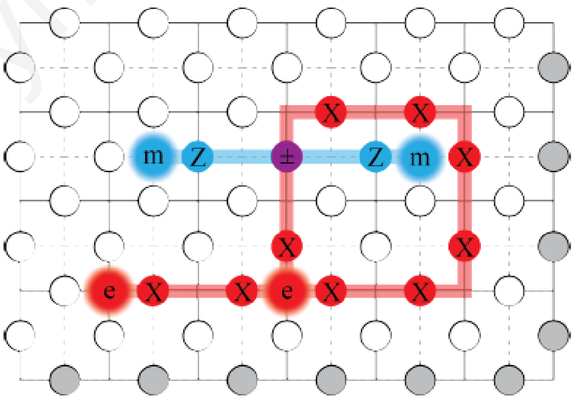

FIG. 5: Moving an e particle around an m particle causes the overall state to pick up a phase of -1, because the X and Z strings anticommute at exactly one site.

total quantum dimension. Therefore, we have $d_i = 1$ for all the four quasiparticles and total dimension is $\mathcal{D} = 2$. All quasiparticles are Abelian ones.

3. From Verlinde formula $a \times b = \sum_c N_{ab}^c c$ where $N_{ab}^c = \sum_m \mathcal{S}_{am}\mathcal{S}_{bm}\mathcal{S}^*_{mc}/\mathcal{S}_{1m}$, we have the Z_2 fusion rule: $1 \times 1 = 1$, $1 \times x = x$, $x \times x = 1$ for $x = s, v$ and $s \times v = sv$.

4. The elements $\mathcal{U}_{ii}$ corresponds to the "topological spin" θ of quasiparticle. Thus we have $\theta_1 = \theta_s = \theta_v = 1$ and $\theta_{sv} = -1$, which show the fermionic self statistics of sv and the bosonic self statistics of $1, s$ and v quasiparticles.

5. The quantity $\mathcal{S}_{22} = 1(\mathcal{S}_{33} = 1)$ shows that s (or v) has the trivial braiding statistics to itself. But $\mathcal{S}_{32} = \mathcal{S}_{23} = -1$ indicates s (v) obeys semionic statistics relative to v (s). which means s (v) will pick up a π phase when it encircles v (s). Similar argument shows that, the π phase is acquired when s (v) encircles sv.

The modular matrices Eq. 13 confirm that the quasiparticles indeed obey Z_2 quasiparticle statistics, including the quasiparticle quantum dimensions, fusion rules and topological spins. Moreover, quasiparticle 1 has trivial statistics related to the other three quasiparticles so that 1 is the identity quasiparticle. Based on the above observations, we determine that the quasiparticle s (v) behaves as a spinon (vison) in Z_2 gauge theory.

APPLICATIONS IN THE QUANTUM COMPUTATION

Fault-tolerant quantum computation

The ground state of the toric-code model is protected from local perturbations. For instance, small deviations from the ideal Hamiltonian will lead to virtual production of pairs of "charges" or "fluxes" which will quickly recombine because of the energy gap. More explicitly, we consider a local σ^x perturbation $V = \lambda \prod_j \sigma_j^x$ where j runs over the spin on the lattice and $\lambda \ll 1$. Local excitation is irrelevant, but we need take care of the topological defects to travel on a non-contractible loop winding around the torus before returning to annihilate with its partner (see Fig. 3). This requires tunneling a distance $\sim N$ and is exponentially small. That is, if we do perturbation theory, the small parameter λ is the strength of the deviations from the ideal Hamiltonian divided by the excitation gap. Processes in which a (virtual) topological defect winds around the torus appear at order λ^N.

Since the ground states are "topologically protected", we can take advantage of the protected ground state for quantum computation. From here on we will use the terms spin and qubit interchangeably, but this is just a matter of terminology. A qubit can be any 2-level quantum system; referring to it as such simply indicates our intention to use this system for information processing. The Hilbert space of n qubits has dimension 2^n, so we can store 2 qubits worth of information in the ground state of the toric code. The properties of our Hamiltonian ensure that this information is distributed across the state of the entire system rather than localized to the states of individual spins; this protects the information from local errors. A scheme for storing quantum information in this way is called a quantum error-correcting code [M. Nielsen and I. L. Chuang, Cambridge Univ. Press (2010)], which is where the name toric code comes from. Toric code is also written as $[L^2, 2, L]$, i.e. the first number is total physical qubits, the second number is the logical bits, and the third one is the typical length of operators to error correction.

It turns out we can do more than just store information. Let us re-label the four orthogonal ground states (a (even number of down-spin along both directions), b, c, d (odd number of down-spin along both directions)) from Fig. 3 as $|00\rangle, |01\rangle, |10\rangle, |11\rangle$, to make the analogy explicit for two logical qubits (as opposed to physical qubits). Then a string of σ^x operators looped around the horizontal dimension maps $|00\rangle$ to $|10\rangle$, and a loop of σ^z operators in the vertical direction maps $|00\rangle$ to $|00\rangle$, and $|10\rangle$ to $-|10\rangle$, therefore we can identify these strings as logical Pauli operators X_1 and Z_1 acting on the first logical qubit. Similarly, the remaining loops of σ^x and σ^z operators act as logical operators X_2, Z_2 on the second qubit.

Note that the action of these logical operators does not depend on a specific path as long as that path winds around the torus, and we could also interpret these operators as anyon pairs which tunnel around the torus before self-annihilating. This is just one example of the connections between topological order, anyons, and quantum computation. A further discussion is beyond the scope of this review, but before we conclude it should be pointed out that the toric code and its close relative the surface code [S. B. Bravyi and A. Y. Kitaev, (1998), arXiv:9811052] underpin some of the leading proposals for fault-tolerant quantum computation [R. Raussendorf, J. Harrington, and K. Goyal, New J. Phys. 9, 199 (2007)]. This section is intended to provide some insight into why that is the case, and to encourage further reading (e.g. D. Browne, "Topological Codes and Computation", is quite accessible).

Quantum error correction

Let us take X-error here, which flip the direction of one local plaquette. The flip this plaguette leads to $P_a = -1$, which is detectable by P_a. At the second step, one attempts to correct errors by pairing two (-1) plaguettes by measuring along the shortest paths. So in general, the (local) errors are detectable and correctable.

Transition

In this section, we consider the toric code model subjected to the transverse field:

$$H = -J\sum_b \sigma_b^z - \lambda \sum_a P_a \tag{13}$$

supplemented with the constraint

$$V_s = 1, \forall s. \tag{14}$$

We will show that this model is Kramers-Wannier dual to the transverse field Ising model.

We define a spin-variable living on the plaguette r^* of the original model:

$$\mu_{r^*}^z \equiv V_{a=r^*} = \prod_{b \in plaquetter^*} \sigma_b^x \tag{15}$$

and another spin-variable is defined along a path γ_{r^*} which is ending at plaquette r^* (see

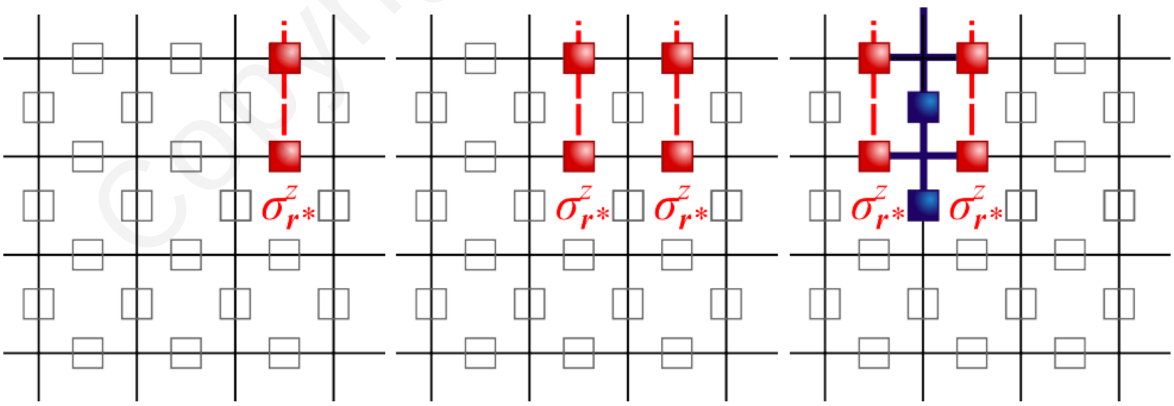

FIG. 6: (Left) The definition of μ_r^x, the red dashed line represents the path γ_{r*}. (Center): Two adjacent strings $\mu_r^x \mu_{r+x}^x$. (Right): Using the gauge constraint, we see that $\mu_r^x \mu_{r+x}^x = \sigma_{r+x/2}^z$.

Fig. 6):

$$\mu^x_{r*} = \prod_{b\in\gamma_{r*}} \sigma^z_b \tag{16}$$

Using these variables and the gauge constraint, we find

$$H = -J \sum_{\langle r^*, r'^* \rangle} \mu^x_{r*}\mu^x_{r'*} - \lambda \sum_{r^*} \mu^z_{r*} \tag{17}$$

The relationship between λ terms is obvious and follows from the definition; The Ising interaction $\mu^z_{r*}\mu^z_{r*+y}$ are

$$\mu^x_{r*}\mu^x_{r*+y} = \prod_{b\in\gamma_{r*}} \sigma^x_b \prod_{b'\in\gamma_{r*}+y} \sigma^x_{b'} = \sigma^x_{r^*+a/2\hat{y}} \tag{18}$$

The Ising interaction $\mu^z_{r*}\mu^z_{r*+x}$ are

$$\mu^x_{r*}\mu^x_{r*+x} = \prod_{b\in\gamma_{r*}} \sigma^z_b \prod_{b'\in\gamma_{r*}+x} \sigma^z_{b'} = \sigma^z_{r^*+a/2\hat{x}} \tag{19}$$

which we have used the Gauss law $V_s = 1, \forall s$.

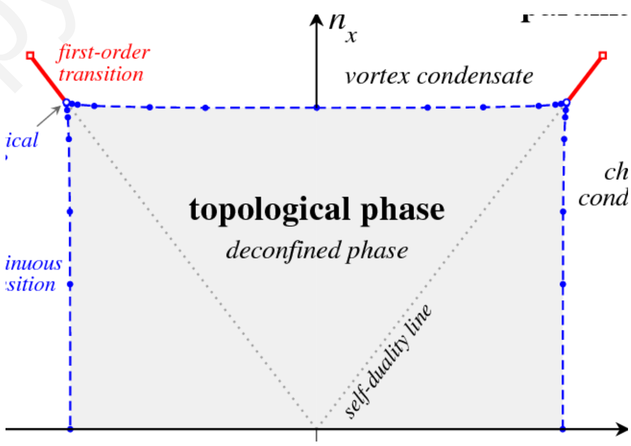

FIG. 7: Phase diagram subjected to the longitudinal field.

* Electronic address: zhuwei@westlake.edu.cn

[1] E. Fradkin, Field Theories of Condensed Matter Physics. Cambridge University Press, 2013.