Lecture note on the Brownian motion

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The nonequilibrium dynamics is currently being addressed along various frontiers in condensed matter physics, bridging from transport through molecules and quantum dots, to the light-control of complex materials, and all the way to the fundamental questions of statistical physics. The typical examples include femtosecond laser pulses triggering ultra-fast phase transitions, and the study of time-evolution of solids on timescales in which their microscopic constituents are not even locally in thermal equilibrium. Moreover, using cold atoms in optical lattices, one can address long-standing questions of nonequilibrium statistical physics: How does a many-body relax to thermal equilibrium? Does a system heat up indefinitely under external time-periodic driving, or does its energy eventually localize? Solving the quantum dynamics of many particles out of equilibrium still remains a challenge, but the intensive research of the previous years has lead to enormous progress in the development of numerical methods.

In this chapter, we will discuss the dynamics of systems that are out of equilibrium and develop tools to describe the processes by which they decay back to equilibrium.

BROWNIAN MOTION

The dynamics of a Brownian particle provides a paradigm for describing equilibrium and nonequilibrium processes. When a relatively massive particle (like a grain of pollen) is immersed in a fluid, it is observed to undergo rapid, random motion, even when it is in thermodynamic equilibrium with the fluid. The agitated motion of the Brownian particle is a consequence of random kicks that it receives from density fluctuations in the equilibrium fluid, and these density fluctuations are a consequence of the discrete (atomic) nature of matter. Thus, Brownian motion provides evidence on the macroscopic scale of the fluctuations that are continually occurring in equilibrium systems.

A phenomenological theory of Brownian motion can be obtained by writing Newton's equation of motion for the massive particle and including a systematic friction force and a randomforce that mimics the effects of the many degrees of freedom of the fluid in which the massive particle is immersed. The equation of motion for the Brownian particle is called the Langevin equation.

Langevin equation

Consider a particle of mass m and radius a, immersed in a fluid of particles of mass m_f ($m_f \ll m$) and undergoing Brownian motion. The fluid gives rise to a retarding force (friction) that is proportional to the velocity, and a randomforce, $\eta(t)$, due to randomdensity fluctuations in the fluid. The equation ofmotion for the Brownian particle can be written

$$m\frac{dv(t)}{dt} = -\gamma v(t) + \eta(t) \tag{1}$$

where $v(t)$ is the velocity of the particle at time t and γ is the friction coefficient. This equation is called the Langevin equation. In mathematics, this equation is ordinary differential equation (ODE) with randomness.

We will assume that $\eta(t)$ is a Gaussian white noise process with zero mean so that

$$\langle \eta(t) \rangle = 0, \langle \eta(t_1)\eta(t_2) \rangle = g\delta(t_1 - t_2). \tag{2}$$

Let us calculate the average displacement and average displacement fluctuation:

$$mx\frac{d^2x}{dt^2} = -\gamma x\frac{dx}{dt} + x\eta(t) \tag{3}$$

$$\rightarrow \frac{m}{2}\frac{d^2}{dt^2}x^2 - m(\frac{dx}{dt})^2 = -\frac{\gamma}{2}\frac{d}{dt}x^2 + x\eta \tag{4}$$

where we used $x\frac{d^2x}{dt^2} = \frac{d}{dt}(x\frac{dx}{dt}) - (\frac{dx}{dt})^2 = \frac{1}{2}\frac{d^2}{dt^2}x^2 - (\frac{dx}{dt})^2$. Next we think about the number of particles is large, and we only consider the averaged value of displacement,

$$\frac{m}{2}\frac{d^2}{dt^2}\overline{x^2} - m\overline{v^2} = -\frac{\gamma}{2}\frac{d}{dt}\overline{x^2} + \overline{x\eta} \tag{5}$$

Since the random force is independent of the particle position, we have

$$\overline{x\eta} = \overline{x\eta} = 0 \tag{6}$$

And we assume the particle m has reached the thermal equilibrium with the media, so we apply the energy equipartition theorem:

$$\frac{1}{2}\overline{mv^2} = \frac{1}{2}k_BT \tag{7}$$

Thus, we have

$$\frac{d^2}{dt^2}\overline{x^2} + \frac{1}{\tau}\frac{d}{dt}\overline{x^2} - \frac{2k_BT}{m} = 0 \tag{8}$$

with $\tau = m/\gamma$. The general solution of this ODE is

$$\overline{x^2} = \frac{2k_BT\tau^2}{m}(\frac{t}{\tau} - (1 - e^{-\frac{t}{\tau}})) \tag{9}$$

where we set $\overline{x^2} = 0$, $\frac{d}{dt}\overline{x^2} = 0$ at $t = 0$.

For the long time limit $t \gg \tau$, we have

$$\overline{x^2} = \frac{2k_BT\tau}{m}t = 2Dt \tag{10}$$

where the diffusion coefficient $D = \frac{k_BT}{\gamma}$. The relationship of variance $\overline{x^2} \sim t$ (but not t^2), was first derived by Einstain.

Let us estimate the time scale $\tau = m/\gamma$. The mass of the particle is $m = \frac{4\pi a^3}{3}\rho$. The random force is from the fiction between media and particle. According to the Stokes theorem, we have $\gamma = 6\pi a\eta$, where η is the viscosity of the media. So we have

$$\frac{\gamma}{m} = \frac{9\eta}{2a^2\rho} \approx \frac{9 \times (1.14 \times 10^{-3}kg/m/s)}{2(10^{-7}m)^2 \times 1 \times 10^3 kg.m^{-3}} \sim 10^7 s^{-1} \tag{11}$$

where we assume the viscosity of water as $1.14 \times 10^{-3}kg/m/s$. So we have $\tau \sim 10^{-7}s$. In the time regime $t > \tau$, we should observe the Brownian motion.

Starting from Eq. 1, one can also get the velocity field as

$$v(t) = v(0)e^{-\frac{\gamma}{m}t} + \frac{1}{m}e^{-\frac{\gamma}{m}t}\int_0^t ds e^{\frac{\gamma}{m}s}\eta(s). \tag{12}$$

So the correlation of the velocity is

$$\begin{aligned}\langle v(t_2)v(t_1)\rangle &= v_0^2 e^{-\frac{\gamma}{m}(t_2+t_1)} + \frac{g}{m^2}\int_0^{t_2} ds_2 \int_0^{t_1} ds_1 \delta(s_2 - s_1) e^{\frac{\gamma}{m}(s_1-t_1)} e^{\frac{\gamma}{m}(s_2-t_2)} \\ &= (v_0^2 - \frac{g}{2m\gamma})e^{-\frac{\gamma}{m}(t_1+t_2)} + \frac{g}{2m\gamma}e^{-\frac{\gamma}{m}|t_1-t_2|}\end{aligned} \tag{13}$$

At $t = 0$, by the equipartition theorem, $\frac{1}{2}m\langle v_0^2\rangle = \frac{1}{2}k_BT$, we estimate $g = 2m\gamma v_0^2$. At long time scale, we have

$$\langle v(t_2)v(t_1)\rangle \approx \frac{k_BT}{m}e^{-\frac{\gamma}{m}|t_1-t_2|} \tag{14}$$

MARKOV PROCESS

Random walk

Here we introduce another type of analysis based on the random walk. We only consider one-dimensional case. Let $x(t)$ be the displacement of particle. And upon the collision, the particle moves a small displacement λ after time Δt. We assume this process is random (with 50% probability move forward). In the total time $t = N\Delta t$, the particle moves N_1 steps forward and $N_2 = N - N_1$ steps afterwards. After N steps, the displacement should be

$$x = (N_1 - N_2)\lambda = m\lambda \tag{15}$$

When N_1 and N_2 is fixed, the number of movements are

$$\frac{N!}{N_1!N_2!} = \frac{N!}{N_1!(N-N_1)!} \tag{16}$$

and

$$\sum_{N_1=0}^{N} \frac{N!}{N_1!(N-N_1)!} = 2^N, \tag{17}$$

so the probability of $x = m\lambda$ is

$$P_N(m) = \frac{\frac{N!}{(\frac{1}{2}(N+m))!(\frac{1}{2}(N-m))!}}{2^N} \overset{N \gg m}{=} \frac{2}{\sqrt{2\pi N}} e^{-m^2/2N} \tag{18}$$

where we used $\ln N! \approx N(\ln N - 1) + \frac{1}{2}\ln(2\pi N)$ and $\ln(N \pm m)/2 \approx \ln N/2 + \ln(1 \pm m/N) \approx \ln N/2 \pm m/N - m^2/2N^2$.

The probability between x and $x + dx$ is

$$P(x)dx = P_N(m)\frac{dx}{2\lambda} = \frac{dx}{\sqrt{2\pi N\lambda^2}} \exp{(-\frac{x^2}{2N\lambda^2})} = \frac{dx}{\sqrt{4\pi Dt}} \exp{(-\frac{x^2}{4Dt})} \tag{19}$$

where $D = \lambda^2/2\Delta t$.

So we have the variance as

$$\overline{x^2} = \int_{-\infty}^{\infty} dx x^2 P(x) = 2Dt \tag{20}$$

which is the same with the result from Langevin equation.

Markov process

A fundamental study of the time evolution of probability distributions is the Markov approximation.

$P_1(y_1, t_1)$ is the **probability density** that the stochastic variable Y has value y_1 at time t_1; $P_{1|1}(y_1, t_1|y_2, t_2)$ is the **conditional probability density** for the stochastic variable Y to have value y_2 at time t_2 given that it has value y_1 at time t_1; $P_n(y_1, t_1; y_2, t_2; \cdots; y_n, t_n)$ is the **joint probability density** that the stochastic variable Y have value y_1 at time t_1, y_2 at time t_2, ..., y_n at time t_n; $P_{k|l}(y_1, t_1; \cdots; y_k, t_k|y_{k+1}, t_{k+1}; \cdots; y_{k+l}, t_{k+l})$ is the **joint conditional probability density** that the stochastic variable Y has values $(y_{k+1}, t_{k+1}; \cdots; y_{k+l}, t_{k+l})$ given that $(y_1, t_1; \cdots; y_k, t_k)$ are fixed.

If the stochastic variable has memory only of its immediate past, which is called the **Markov process**, the joint conditional probability density must have the form

$$P_{n-1|1}(y_1, t_1; \cdots; y_{n-1}, t_{n-1}|y_n, t_n) = P_{1|1}(y_{n-1}, t_{n-1}|y_n, t_n) \tag{21}$$

where $t_1 < t_2 < \cdots < t_n$. The conditional probability density $P_{1|1}(y_{n-1}, t_{n-1}|y_n, t_n)$ in this identity is called the **transition probability**. The Markov character is exhibited by the fact that the probability of the two successive steps is the product of the probability of the individual steps. The successive steps are statistically independent. Two quantities $P(y_1, t_1)$ and $P_{1|1}(y_1, t_1|y_2, t_2)$ completely determine the evolution of a Markov chain. The time evolution of such processes is governed by the **master equation**

$$\frac{\partial P_1(n, t)}{\partial t} = \sum_m [P_1(m, t)W_{m,n} - P_1(n, t)W_{n,m}] \tag{22}$$

which gives the rate of change of the probability $P_1(n, t)$ due to transitions into the state n from all others states (first term on the right) and due to transitions out of state n into all others states (second term on the right). Here, one assume that stochastic variable Y has discrete realizations $\{y(n)\}$ and the transition matrix $W_{m,n}$ is independent of time.

For the Brownian motion, we assume transfer matrix as $W(x, x')$. If $W(x, x')$ changes a function of distance between $x - x'$. We write it as $W(x', x) = W(x'; x - x') = W(x', \xi)$.

The master equation above changes to

$$\frac{\partial P_1(x,t)}{\partial t} = \int dx'[P_1(x',t)W(x',x) - P_1(x,t)W(x,x')]$$
$$= \int d\xi W(x,\xi)P_1(x,t)d\xi - \int d\xi\xi\frac{\partial}{\partial y}[W(x,\xi)P_1(x,t)] + \frac{1}{2}\int d\xi\xi^2\frac{\partial^2}{\partial y^2}[W(x,\xi)P_1(x,t)] + ... - \int dx'P_1($$
$$= -\frac{\partial}{\partial x}[\alpha_1(x)P_1(x,t)] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[\alpha_2(x)P_1(x,t)] \tag{23}$$

where $\alpha_n(x) = \int d\xi\xi^n W(x,\xi)$. For the Brownian problem, we have a symmetry $\alpha_1 = 0$. $\alpha_2 = \langle\xi^2\rangle$. This form is Fokker-Plank equation.

Next we also expand the $P_1(x,t+\tau) = P_1(x,t) + \tau\frac{\partial P_1}{\partial t} + ...$, we have

$$\frac{\partial P_1(x,t)}{\partial t} = \frac{\langle\xi^2\rangle}{2\tau}\frac{\partial^2}{\partial x^2}P_1(x,t) \equiv D\frac{\partial^2}{\partial x^2}P_1(x,t) \tag{24}$$

This is the diffusion equation satisfied by the probability.

Diffusion equation

We can also understand the process from the viewpoint of diffusion equation:

$$\nabla^2 P(x,t) - \frac{1}{D}\frac{\partial P(x,t)}{\partial t} = 0 \tag{25}$$

and then we can solve it as

$$P(x,t) = \frac{N}{(4\pi Dt)^{1/2}}e^{-\frac{x^2}{4Dt}} \tag{26}$$

where N is determined by the renormalization condition $\int_{\infty}^{-\infty} dxn(x,t) = N$.

Using this distribution function, we have

$$\overline{x(t)} = 0, \overline{x^2(t)} = \int_{\infty}^{-\infty} dxP(x,t)x^2 = 2Dt \tag{27}$$

FOKKER-PLANCK EQUATION

The Fokker-Planck equation is the equation governing the time evolution of the probability density for the Brownian particle. It is a second-order differential equation and is exact for the case when the noise acting on the Brownian particle is Gaussian white noise. The derivation of the Fokker-Planck equation is a two step process.

Let us obtain the probability to find the Brownian particle in the interval $x \to x+dx$ and $v \to v+dv$ at time, t. We will consider the space of coordinates, $X=(x,v)$, (x and v being the displacement and velocity of the Brownian particle, respectively), where $-\infty < x < \infty$ and $-\infty < v < \infty$. The Brownian particle is located in the infinitesimal area, $dxdv$, with probability $\rho(x,v,t)dxdv$. Since the Brownian particle must lie somewhere in this space, we have the condition

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dv \rho(x,v,t) = 1 \tag{28}$$

Let us now consider a fixed finite area A_0 in phase space. The probability to find the Brownian particle in this area is $P(A_0) = \int_{A_0} dxdv\rho(x,v)$. Since the Brownian particle cannot be destroyed, any change in the probability contained in A_0 must be due to a flow of probability through the sides of A_0. Thus,

$$\frac{\partial P(A_0)}{\partial t} = \frac{\partial}{\partial t}\int_{A_0} dxdv\rho(x,v) = -\oint_{L_0} \rho(x,v,t)\dot{X}\cdot dS_0 \tag{29}$$

where dS_0 denotes a differential surface element along the edge of area A_0, $\rho\dot{X}$ is the probability current through the edge, and L_0 is the line around the edge of area element, A_0. Using the Gauss's theorem, $\oint_{L_0} \rho(x,v,t)\frac{d}{dt}X\cdot dS_0 = \int_{A_0} dxdv\nabla_X(\dot{(X)}\rho)$, we find

$$\frac{\partial}{\partial t}\int_{A_0} dxdv\rho(x,v,t) = -\int_{A_0} dxdv\nabla_X(\dot{X}\rho(x,v,t))$$

$$\Rightarrow \frac{\partial}{\partial t}\rho(x,v,t) = -\nabla_X(\dot{X}\rho(x,v,t)) = -\frac{\partial(\dot{x}\rho)}{\partial x} - \frac{\partial(\dot{v}\rho)}{\partial v}$$

$$\frac{\partial}{\partial t}\rho(x,v,t) = -L_0\rho(t) - L_1(t)\rho \tag{30}$$

$$L_0 = v\frac{\partial}{\partial x} - \frac{\gamma}{m} - \frac{\gamma}{m}v\frac{\partial}{\partial v}, L_1 = \frac{1}{m}\eta(t)\frac{\partial}{\partial v} \tag{31}$$

where we used the Langevin equation $\frac{dv(t)}{dt} = -\frac{\gamma}{m}v(t) + \frac{1}{m}\eta(t)$.

When we observe an actual Brownian particle we are observing the average effect of the random force on it.Therefore, we introduce an observable probability, $P(x,v,t)dxdv$, to find the Brownian particle in the interval $x \to x+dx$ and $v \to v+dv$. We define this observable probability to be $P(x,v,t) = \langle\rho(x,v,t)\rangle_\eta$. Since the random force, $\eta(t)$, has zero mean and is a Gaussian white noise, the derivation of $P(x,v,t)$ is straightforward and very instructive. It only takes a bit of algebra. We first introduce a new probability density, $\sigma(t)$, such that $\rho(t) = e^{-L_0 t}\sigma(t)$. Inserting it into Eq. 30, we have

$$\frac{\partial}{\partial t}\sigma(t) = -V(t)\sigma(t), V(t) = e^{L_0 t}L_1(t)e^{-L_0 t} \tag{32}$$

This equation has the formal solution

$$\sigma(t) = \exp(\int_0^t dt' V(t;))\sigma(0) = [\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}(\int_0^t dt' V(t'))^n]\sigma(0) \tag{33}$$

We now can take the average. Because the noise has zero mean and is Gaussian, Wicks theorem applies (let us ponspone it to later). Only even values of n will remain,

$$\langle\sigma(t)\rangle = [\sum_{n=0}^{\infty} \frac{1}{2n!}(\int_0^t dt' V(t'))^{2n}]\sigma(0) = [\sum_{n=0}^{\infty} \frac{1}{n!}(\frac{1}{2}\int_0^t dt_1 \int_0^t dt_2 \langle V(t_2)V(t_1)\rangle)^n]\sigma(0) \tag{34}$$

The integral inside is

$$\frac{1}{2}\int_0^t dt_1 \int_0^t dt_2 \langle V(t_2)V(t_1)\rangle = \frac{g}{2m^2}\int_0^t dt_1 \int_0^t dt_2 \delta(t_2 - t_1) e^{L_0 t_2} \frac{\partial}{\partial v} e^{-L_0(t_2-t_1)} \frac{\partial}{\partial v} e^{-L_0 t_1} \tag{35}$$

$$= \frac{g}{2m^2}\int_0^t dt_1 e^{L_0 t_1} \frac{\partial^2}{\partial^2 v} e^{-L_0 t_1} \tag{36}$$

Taking the time derivative on the above equation

$$\frac{\partial}{\partial t}\langle\sigma(t)\rangle = \frac{g}{2m^2} e^{L_0 t} \frac{\partial^2}{\partial^2 v} e^{-L_0 t}\langle\sigma(t)\rangle \tag{37}$$

With this result, we can obtain the equation of motion of $P(x, v, t) = \langle\rho(x, v, t)\rangle$. Let us note that $\langle\rho(t)\rangle = e^{-L_0 t}\langle\sigma(t)\rangle$ and take the derivative of $\langle\rho(t)\rangle$ with respect to time, t. We then obtain

$$\frac{\partial}{\partial t}\langle\rho(t)\rangle = -L_0\langle\rho(t)\rangle + e^{-L_0 t}\frac{\partial}{\partial t}\langle\sigma(t)\rangle = -L_0\langle\rho(t)\rangle + \frac{g}{2m^2}\frac{\partial^2}{\partial^2 v}\langle\rho(t)\rangle \tag{38}$$

$$\frac{\partial}{\partial t}P = -v\frac{\partial}{\partial x}P + \frac{\partial}{\partial v}(\frac{\gamma}{m}P) + \frac{g}{m^2}\frac{\partial^2}{\partial^2 v}P \tag{39}$$

This is the FokkerPlanck equation for the observable probability, $P(x, v, t)dxdv$, to find the Brownian particle in the interval $x \to x + dx$ and $v \to v + dv$ at time, t.

Next we consider the case of strong fiction limit, i.e. the particle motion is only determined by the random force $\eta(t)$, the Langevin equation becomes $\frac{dv(t)}{dt} = -\frac{\gamma}{m}v(t) + \frac{1}{m}\eta(t) = 0 \to v(t) = \frac{\eta(t)}{\gamma}$. In this case, the master equation reduces to

$$\frac{\partial}{\partial t}\rho(x, v, t) = -\nabla_X(\dot{X}\rho(x, v, t)) = -\frac{\partial(\dot{x}\rho)}{\partial x} - \frac{\partial(\dot{v}\rho)}{\partial v} \tag{40}$$

$$\frac{\partial}{\partial t}\rho(x, v, t) = -\frac{\partial(\dot{x}\rho)}{\partial x} = -L_0\rho(t) - L_1(t)\rho \tag{41}$$

$$L_0 = 0, L_1 = \frac{1}{m}\eta(t)\frac{\partial}{\partial x} \tag{42}$$

And then we obtain the equation like

$$\nabla^2 P(x, t) - \frac{1}{D}\frac{\partial P(x, t)}{\partial t} = 0 \tag{43}$$

where $D = g/2\gamma^2 = k_B T/\gamma$. This equation is also called Diffusion equation.

FLUCTUATION-DISSIPATION THEOREM

Let us recall the Langevin equation,

$$\frac{dv(t)}{dt} = -\frac{v(t)}{\tau} + \frac{\eta(t)}{m} \tag{44}$$

To solve it, we assume $v(t) = f(t)e^{-t/\tau}$, and obtain

$$\frac{df}{dt} = e^{t/\tau}\frac{\eta(t)}{m} \tag{45}$$

and

$$v(t) = v(0)e^{-t/\tau} + e^{-t/\tau}\frac{1}{m}\int_0^t e^{\xi/\tau}\eta(\xi)d\xi \tag{46}$$

The velocity satisfies $\overline{v(t)} = v(0)e^{-t/\tau}$. So the velocity decays to zero in the long time. The information of initial state is washed out.

We can obtain the velocity autocorrelation function. If we make use of the fact that $\langle v(0)\eta(t)\rangle = 0$, then we can write

$$\langle v(t_2)v(t_1)\rangle = v^2(0)e^{-(t_1+t_2)/\tau} + 1/m^2\int_0^{t_2}\int_0^{t_1} e^{(\xi_1+\xi_2)/\tau}e^{-t_1/\tau-t_2/\tau}\langle\eta(t_1)\eta(t_2)\rangle d\xi_1 d\xi_2 \tag{47}$$

$$= v^2(0)e^{-(t_1+t_2)/\tau} + 1/m^2\int_0^{t_1}\int_0^{t_2} e^{(\xi_1-t_1)/\tau}e^{(\xi_2-t_2)/\tau}g\delta(\xi_1-\xi_2)d\xi_1 d\xi_2 \tag{48}$$

$$= \begin{cases} (v^2(0) - \frac{g}{2m\gamma})e^{-\gamma(t_2+t_1)/m} + \frac{g}{2m\gamma}e^{-\gamma(t_2-t_1)/m}, t_2 > t_1 \\ (v^2(0) - \frac{g}{2m\gamma})e^{-\gamma(t_2+t_1)/m} + \frac{g}{2m\gamma}e^{-\gamma(|t_2-t_1|)/m}, t_2 < t_1 \end{cases} \tag{49}$$

If the Brownian particle is in equilibrium, its velocity autocorrelation function must be stationary and can only depend on time differences $t_1 - t_2$. Therefore, the first term $t_1 + t_2$ should vanish. In the condition of $t \to \infty$, $\overline{v^2(t)}$ should approach the thermal condition by the equipartition theorem: $mv^2(t)/2 = k_BT/2$. Thus $g = 2\gamma k_BT$:

$$\langle v^2(t)\rangle = \frac{k_BT}{m}e^{-2t/\tau} \tag{50}$$

Since $g = 2\gamma k_BT$, we have

$$\gamma = \frac{1}{2k_BT}g = \frac{1}{2k_BT}\int_{-\infty}^{\infty}\langle\eta(0)\eta(t)\rangle dt \tag{51}$$

This is the fluctuation-dissipation theorem. The left side is viscosity γ , which is the dissipation property of the system. The right side is time-correlation function of the random force, which the thermal fluctuation property of random force. This formula connects the fluctuation with the dissipation.

Fluctuations about the equilibrium state decay on the average according to the same macroscopic laws that govern the decay of a nonequilibrium system to the equilibrium state. If we can probe equilibrium fluctuations, we have a means of probing the decay processes in a system. Linear response theory provides a tool for probing equilibrium fluctuations by applying a weak external field which couples to the system. The system responds to the field in a manner that depends entirely on the spectrum of the equilibrium fluctuations. The response to the dynamic field is measured by the susceptibility matrix. The fluctuationdissipation theorem links the susceptibility matrix to the correlation matrix for equilibrium fluctuations. According to the fluctuationdissipation theorem, the spectrum of equilibrium.

EPIDEMIC MODEL

In 2020-2022, the world has been gripped by a pandemic, COVID 19 virus. The coronavirus pandemic is undoubtedly the greatest challenge the world has faced in over a generation. The statistical modelling is not only to manage the short-run crisis for the health services, but also to explain the pandemics course and establish the effectiveness of different policies.

The so-called SIR model has been used as the basic building block of epidemiological modelling. The total (initial) population N is categorised in five groups, namely, the susceptible (healthy without infection) $S(t)$, the exposed $E(t)$, the infected $I(t)$, died $D(t)$ and the recovered $R(t)$, where t is the time variable. The governing differential equations of the model are:

$$\frac{dD(t)}{dt} = \alpha I(t) \tag{52}$$

$$\frac{dR(t)}{dt} = \gamma I(t) \tag{53}$$

$$\frac{dS(t)}{dt} = -\beta S(t) \times (I(t)/N) \tag{54}$$

$$\frac{dE(t)}{dt} = \beta S(t) \times (I(t)/N) - \delta E(t) \tag{55}$$

$$\frac{dI(t)}{dt} = \delta E(t) - \alpha I(t) - \gamma I(t) \tag{56}$$

where α is virus induced average fatality rate, γ is recovery rate of infected individuals (the reciprocal is the infection period), β is probability of disease transmission per contact per unit time, δ is rate of progression from exposed to infected (the reciprocal is the incubation period).

Let us estimate the death rate by the published data. The death rate in Hong Kong is around 25 per 100,000 residents. So we set $\alpha \approx 0.025\%$. This rate is higher than the reports in UK. There is a statistical data for different ages in Fig. 1.

The incubation period is around 10 days, according to various estimations. So we choose $\delta = 1/10 = 0.1$. The infection period is around one week, so we set $\gamma = (1 - \alpha)/7$. β is the parameter that is hard to estimate, which depends on many facts. We can choose various parameter to simulate, see Fig. 2.

- Death is always neglecting small, so most of people donot have to worry about it;

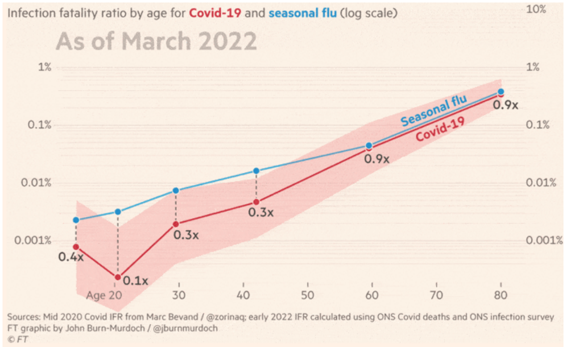

FIG. 1: Death rate for different age groups.

- The key fact is probability of disease transmission β. We wish vaccine can significantly reduce β. Or, one can keep the activities to the limited level to control β. The smaller β, the more muted the epidemic change;
- For a large β, say $\beta \approx 50\%$ (i.e. For an individual, it has 50% probability to catch virus per contact everyday), there is a peak of infection coming in 1-2 month. We need to beware of a medical run then.

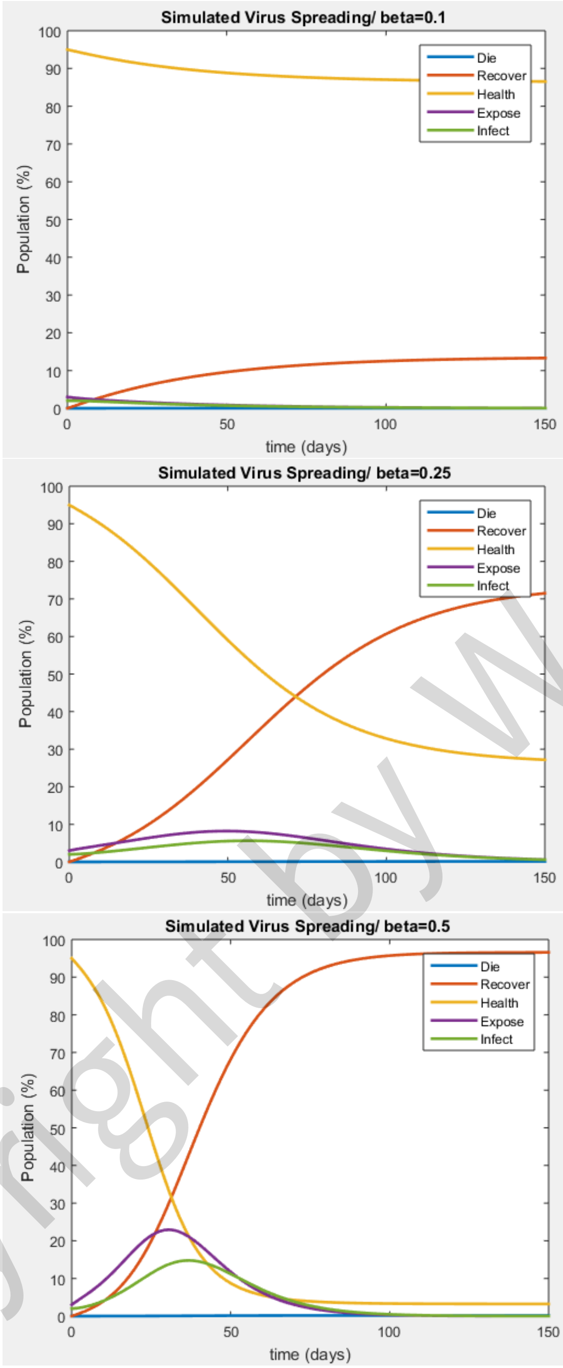

FIG. 2: Evolution of population in virus spreading. (Left) $\beta = 0.1$; (Middle) $\beta = 0.25$; (Right) $\beta = 0.50$. Here we set the death rate $\alpha = 0.025\%$, rate of progression from exposed to infected $\delta = 0.10$, recovery rate of infected individuals $\gamma = (1 - \alpha)/7$. $\beta \in [0, 1]$ is probability of disease transmission per contact per unit time.