Notes on the Ising model

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Contents

ISING MODEL

The Ising model was originally introduced by Lenz in 1920, to describe the transition from a para- to a ferromagnetic phase in a magnetic lattice. The solution was given by Ising (1925) in dimensions $d = 1$. Later, it has become a paradigm for several different systems, including binary alloy, lattice gases, and large biological molecules (Huang's textbook). The reason of its relevance and popularity resides on the fact that it accounts for an order-disorder transition on a lattice by dealing with a minimum number of variables and external parameters. Beyond $d = 1$, the exact solution are available in two dimension $d = 2$, first in a vanishing external field [Onsager Phys. Rev. 65, 117 (1944); Kaufman Phys. Rev. 76, 1232 (1949)], and then in a nonzero external field (Yang, 1952). This allows to extract all details of the model, including its critical exponents, which can then be compared with approximate or numerical estimates of similar models. Actually, in quite a long time, it is the only non-trivial example of a phase transition that can be worked out with mathematical rigor. In three and more dimensions, a mean-field approximation is still capable of grasping most of the features of the Ising model.

Consider a d-dimensional lattice with N sites, and assume that the state of each lattice site, labeled by i, with $i = 1, ..., N$, can be characterized by the value of a single variable, say σ_i, taking only the possible values $\sigma_i = \pm 1$. For the sake of definiteness, we might think of magnetic spins residing on such sites, with $\sigma_i = 1$ corresponding to a spin up, and $\sigma_i = -1$ corresponding to a spin down. The Ising model is then a minimal model allowing for interaction between spins residing at nearest-neighbour sites in the lattice, and for spins with an external magnetic field, B say. The Hamiltonian (i.e. the classical energy) of the model is then given by

$$H = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j - B \sum_i \sigma_i \tag{1}$$

Here, $-J$ is the interaction energy between sites i and j, and the summation restricts that to nearest-neighbouring sites only. In the following, for the sake of simplicity, we shall assume $-J < 0$. The number of nearest neighbours, or coordination number, z, is determined by the geometry of the lattice, being $z = 2d$ for a cubic lattice in d-dimensions.

The Ising model differs from the Heisenberg model in that the spins are purely classical.

They do not obey quantum commutation relations as do the spins in the Heisenberg model.

The partition function (in canonical ensemble) of the Ising model can be written as

$$Z = \sum_{\sigma_i=\pm,\ldots,\sigma_N=\pm} e^{-\beta H} \tag{2}$$

where the summation is over all possible values of spins. According to the laws of statistical mechanics, the partition function determines thermodynamic properties. For example, the thermal dynamic functions are obtained in the usual manner from the free energy:

$$F = -\beta \ln Z \tag{3}$$

Some other quantities such as the specific heat can be obtained by

$$C = \frac{\partial U}{\partial T}, U = -kT^2 \frac{\partial F}{\partial T} \tag{4}$$

The thermal average of the magnetization $M = \sum_i \sigma_i$ can be extracted from the partition function as

$$\langle M \rangle = \langle \sum_i \sigma_i \rangle = k_B T \frac{\partial}{\partial B} \log Z \tag{5}$$

The central interest about Ising model is the phase transition from an ordered state to a disordered state (details will be discussed below). Above the critical temperature T_c the system is in a disordered state, which corresponds to a random distribution of the spin values. Below the critical temperature T_c (nearly) all spins are aligned, even in the absence of an external applied magnetic field H. If we heat up a cooled ferromagnet, the magnetization vanishes at T_c and the ferromagnet switches from an ordered to a disordered state. This is a phase transition of second order. This will be the main topic in this course.

EXACT SOLUTION IN ONE-DIMENSION $d = 1$

For $d = 1$ and periodic boundary condition, the Ising Hamiltonian reduces to

$$\begin{aligned} H &= -J \sum_{\langle ij \rangle} \sigma_i \sigma_j - B \sum_i \sigma_i \\ &= -J \sum_i \sigma_i \sigma_{i+1} - \frac{B}{2} \sum_i (\sigma_i + \sigma_{i+1}) \end{aligned} \tag{6}$$

and the partition function becomes

$$
\begin{aligned}
Z &= \sum_{\sigma_1=\pm,\ldots,\sigma_N=\pm} \exp[\beta \sum_i J\sigma_i\sigma_{i+1} + B/2(\sigma_i + \sigma_{i+1})] \\
&= \sum_{\sigma_1=\pm,\ldots,\sigma_N=\pm} \exp[\beta J\sigma_1\sigma_2 + B/2(\sigma_1 + \sigma_2)] \exp[\beta J\sigma_2\sigma_3 + B/2(\sigma_2 + \sigma_3)] \ldots \exp[\beta J\sigma_N\sigma_{N+1} + B/2(\sigma_N \\
&= \sum_{\sigma_1=\pm,\ldots,\sigma_N=\pm} \langle\sigma_1|T|\sigma_2\rangle\langle\sigma_2|T|\sigma_3\rangle\ldots\langle\sigma_{N-1}|T|\sigma_N\rangle\langle\sigma_N|T|\sigma_1\rangle
\end{aligned} \tag{7}
$$

where we introduce the transfer matrix operator T defined as

$$
\langle\sigma_i|T|\sigma_{i+1}\rangle = \exp[\beta J\sigma_i\sigma_{i+1} + B/2(\sigma_i + \sigma_{i+1})] = \begin{pmatrix} e^{\beta(J+B)} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta(J-B)} \end{pmatrix}_{\sigma_i\sigma_{i+1}} \tag{8}
$$

Using the property of trace, we have

$$
Z = \sum_{\sigma_1=\pm} \langle\sigma_1|T^N|\sigma_1\rangle = Tr[T^N] = \lambda_+^N + \lambda_-^N = \lambda_+^N(1 + (\lambda_-/\lambda_+)^N) \to \lambda_+^N \tag{9}
$$

where $\lambda_\pm$ are eigenvalues of T:

$$
\lambda_\pm = e^{\beta J}[\cosh(\beta B) \pm \sqrt{\cosh^2(\beta B) - 2e^{-\beta J}\sinh(2\beta J)}] \tag{10}
$$

In the thermodynamic limit, $N \to \infty$, only the largest eigenvalue, λ_+ contributes to the partition function, and all thermodynamic function, e.g. the free energy per site

$$
\begin{aligned}
f &= -k_B T \lim_{N\to\infty} \frac{1}{N} \log Z \\
&= -k_B T \log \lambda_+ = -J - k_B T \log[\cosh(\beta B) + \sqrt{\cosh^2(\beta B) - 2e^{-\beta J}\sinh(2\beta J)}]
\end{aligned} \tag{11}
$$

The average magnetization per site is then given by

$$
M = -\frac{\partial f}{\partial B} = \frac{\sinh \beta B}{\sqrt{\sinh^2(\beta B) + 4e^{4\beta J}}} \tag{12}
$$

Since $M = 0$ as $B \to 0$, the order parameter does not form spontaneously, and thus the Ising model is not characterized by any phase transition in d $= 1$ dimensions. One can check that, at zero temperature $T = 0$, $M = 1$, which means there is ordered phase at zero temperature, so $T_c = 0$.

$\diamond$ *H*omework: Please calculate the spin-spin correlation function $C(r) = \langle\sigma_i\sigma_{i+r}\rangle$ of 1d Ising model, in the case of $B = 0$.

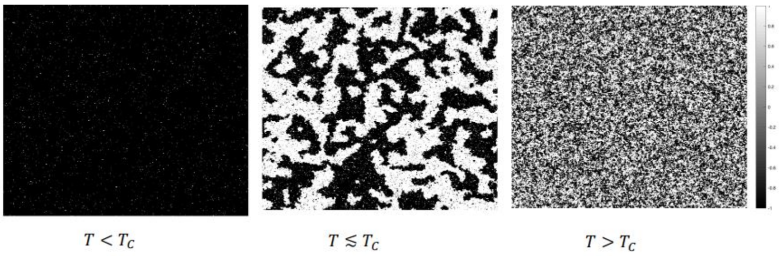


FIG. 1: Solution 2d Ising model using large-scale Monte Carlo simulation.

EXACT SOLUTION OF ISING MODEL IN TWO-DIMENSION $d = 2$

Mapping from two-dimension classical Ising model to quantum transverse Ising model

Let us move to study the case in 2d. The result will be different from the case in 1d. We start with the partition function again

$$Z = \sum_{\sigma_1=\pm,\ldots,\sigma_N=\pm} \exp[-\beta H], H = -J\sum_{\langle ij\rangle}\sigma_i\sigma_j \tag{13}$$

Since we are dealing with the model in two dimension (squared mesh), it is better to introduce coordinates (p, q) for each site, where $p, q \in Z$, and denote the coordinates in x- and y-direction, respectively. We generalize our model allowing for different couplings along the x- and y-directions, and having N sites in the x-direction and M-sites in the y-direction but keeping periodic boundary conditions (p.b.c) along both directions. With all these changes, the Hamiltonian can be written as follows:

$$H = \sum_{q=1}^{M} L(q, q+1) \tag{14}$$

$$L(q, q+1) = \sum_{p=1}^{N}(-J_x\sigma_{p,q}\sigma_{p+1,q} - J_y\sigma_{p,q}\sigma_{p,q+1}) \tag{15}$$

We consider first the case $J_x = 0$, that corresponds to N decoupled one-dimensional Ising models. Let us consider one of those chains, say the p-th one and at a given site q. Then, the partition function consists of a product of terms as follows,

$$Z_p = \sum_{\sigma_{p,1},\ldots,\sigma_{p,M}} \prod T^y_{p,q}, \quad T^y_{p,q} = e^{\beta J_y\sigma_{pq}\sigma_{p,q+1}} \tag{16}$$

Since the variables $\sigma_{p,q}$ have two possible values, we can represent them by a two component vector (a spinor):

$$\sigma_{p,q} = 1 = (1,0)^T, \sigma_{p,q} = -1 = (0,1)^T \tag{17}$$

such that the transfer matrix T^y_{pq} is the same as that in 1d case discussed in the previous section:

$$T^y_{\sigma_{pq},\sigma_{p,q+1}} = \begin{pmatrix} e^{\beta J_y} & e^{-\beta J_y} \\ e^{-\beta J_y} & e^{\beta J_y} \end{pmatrix}_{\sigma_{pq},\sigma_{p,q+1}} \tag{18}$$

Since a 2×2 matrix can be written in terms of Pauli matrices, we have

$$T^y = e^{\beta J_y} I + e^{-\beta J_y} \hat{\sigma}^x = e^{\beta J_y}(1 + e^{-2\beta J_y} \hat{\sigma}^x) \tag{19}$$

(Here transfer matrix T^y should be understood by $\langle \sigma_{p,q+1} | T^y | \sigma_{p,q} \rangle = e^{\beta J_y \sigma_{pq} \sigma_{p,q+1}}$.)

At this point we recall that T^y_{pq} is part of a partition function, and therefore, it would be easier to interpret what we have, if we could express it as the exponential of an operator. Since

$$e^{a\hat{\sigma}^x} = \cosh a + \sinh a \hat{\sigma}^x = \cosh a(1 + \tanh a \hat{\sigma}^x) \tag{20}$$

we can set $\tanh a = e^{-2\beta J_y}$, so we obtain

$$T^y = (\sinh a \cosh a)^{-1/2} \exp[a\hat{\sigma}^x] = (2\sinh(2J_y))^{1/2} \exp[a\hat{\sigma}^x] \tag{21}$$

where we used the relation

$$\tanh a = e^{-2J_y}, \cosh^2 a - \sinh^2 a = 1 \to \cosh^2 a = \frac{1}{1 - e^{-4J_y}} \tag{22}$$

Until now, we were discussing the one-dimensional Ising model. The corresponding partition function is

$$Z_p = \sum_{\sigma_{p,1},\ldots,\sigma_{p,M}} T^y_{\sigma_{p,1},\sigma_{p,2}} T^y_{\sigma_{p,2},\sigma_{p,3}} \ldots T^y_{\sigma_{p,M},\sigma_{p,1}} = Tr[(T^y)^M] \tag{23}$$

where T^y is the transfer matrix along y-direction. Since the trace is invariant under a unitary transformation, it is more informative to look at the trace after diagonalizing T^y, similar to the discussion in one-dimension case.

For the two-dimensional case, we have still to switch on J_x. Please note that, in above we have introduced matrix $\hat{\sigma}_x$ and defined the basis of $|\sigma_{p,q} = \pm 1\rangle$. Next we can consider two columns p and $p+1$, and we require the transfer matrix T^x satisfying

$$\langle \sigma_{p,q} | T^x | \sigma_{p+1,q} \rangle = e^{\beta J_x \sigma_{p,q} \sigma_{p+1,q}} \tag{24}$$

Here we see that the operator T^x should be such that its matrix elements contain no information on the states at $q+1$. One notices that the following form meets the requirements:

$$T^x_{pq} = \exp[\beta J_x \hat{\sigma}^z_{pq} \hat{\sigma}^z_{p+1,q}] \tag{25}$$

With the results above we arrive at the partition function for the whole system

$$Z = (2\sinh 2J_y)^{NM/2} Tr[T^M] \tag{26}$$

$$T = \exp[J_x \sum_p \hat{\sigma}^z_p \hat{\sigma}^z_{p+1}] \exp[a \sum_p \hat{\sigma}^x_p] \tag{27}$$

This is now the transfer matrix for the two-dimensional anisotropic Ising model. In contrast to the one-dimensional case, we have now instead of a 2×2 matrix, a $2^N \times 2^N$ dimensional array. It is however possible to solve the problem exactly, by means of a Jordan-Wigner transformation making fermions out of spins.

We first notice that the two exponentials $(\hat{\sigma}^x, \hat{\sigma}^z)$ do not commute with each other. This makes in fact the problem really quantum mechanical. Performing these replacements, we can write

$$Z \sim Tr e^{-\beta H}, H = -J \sum_p \sigma^z_p \sigma^z_{p+1} - h \sum_p \sigma^x_p \tag{28}$$

This is the one-dimensional Hamiltonian of the Ising model with transverse field, that due to the presence of two noncommuting pieces is a genuinely quantum mechanical model. Please note that, no thermal fluctuations in the new model, but the quantum fluctuations appear. In this regarding, the above mapping provides an example the equivalence between $d+1$-dimensional classical model and $d-$dimensional quantum model.

The so-called transverse Ising model is a canonical model in the study of quantum critical point or quantum phase transition. It is the simplest model with a quantum critical point. It is exactly solvable, as we will show below. It also connects with the conformal field theory, so it is really very important.

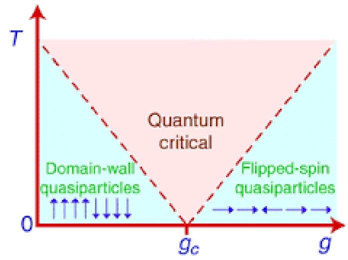


FIG. 2: Phase diagram of 1+1D transverse Ising model.

Phase diagram

In the limit of $h \to 0$, the ground state is an Ising ferromagnet that spontaneously breaks Z_2 symmetry. The two-fold degenerate ground states are

$$|\Psi_\downarrow\rangle = |\downarrow_1 \downarrow_2 \cdots\rangle, |\Psi_\uparrow\rangle = |\uparrow_1 \uparrow_2 \cdots\rangle. \tag{29}$$

When $h \to \infty$, the ground state is a trivial paramagnet that preserves Ising symmetry,

$$|\Psi_x\rangle = |+_1 +_2 \cdots\rangle, + = \uparrow - \downarrow. \tag{30}$$

The above analysis matches the Monte Carlo simulations as shown in Fig. 1. The global phase diagram of the model is easy to image. The interesting problem is, where is the transition point?

Duality

There is a duality transformation which defines new Pauli operators in a dual lattice

$$\tau_i^x = \sigma_i^z \sigma_{i+1}^z, \tau_i^z = \prod_{j \le i} \sigma_i^x \tag{31}$$

then these τ_i^x and τ_i^z satisfy the same commutation and anti-commutation relations of σ_i^x and σ_i^z, i.e. $\{\tau_i^a, \tau_j^b\} = 2\delta^{ab}$. And the original Hamiltonian can be written in terms of $\tau^{x,z}$ as

$$H = -J \sum_p \tau_p^x - h \sum_p \tau_{p+1}^z \tau_p^z \tag{32}$$

where we used the condition that $\tau_{p-1}^z \tau_p^z = \prod_{j<p} \sigma_i^x \prod_{i<p+1} \sigma_i^x = \sigma_p^x$

Since these two Hamiltonian take the same algebra, they should be the same (which means energy spectra, eigenvalues are all the same). But, we notice that the parameter exchange: $J \leftrightarrow h$. This is called duality. In this case, if there is a phase transition point, it should satisfy the condition of (by setting $J = 1$)

$$h_c = \frac{1}{h_c} \Rightarrow h_c = 1 \tag{33}$$

$\diamond$ *Homework:* Please prove the commutation relation between τ_i^x and τ_i^z.

The key feature is, we got a phase transition point $h_c \neq 0$ in 2d, which is quite different from that of 1d. The global phase diagram can be mapped out easily (see Fig. 2).

Diagonalize quantum transverse Ising model

Next we consider the 1+1 D transverse Ising chain with periodic boundary condition. The stategy is to use the fermionic representative [Two-dimensional Ising model as a soluble problem of many fermions, T. D. Schultz, D. C. Mattis, E. H. Lieb]. Here we make the Jordan-Wigner transformation

$$c_n = \frac{\sigma_n^x + i\sigma_n^y}{2} \prod_{m<n} \sigma_m^z, c_n^\dagger = \frac{\sigma_n^x - i\sigma_n^y}{2} \prod_{m<n} \sigma_m^z, \tag{34}$$

$$\sigma_n^+ = \prod_{m<n} (1 - 2c_m^\dagger c_m) c_n, \sigma_n^- = \prod_{m<n} (1 - 2c_m^\dagger c_m) c_n^\dagger, \sigma_n^z = 1 - 2c_n^\dagger c_n \tag{35}$$

The string $\prod_{m<n}(1 - 2c_m^\dagger c_m)$ takes values ± 1, depending on even/odd number of fermions on the left side of n. One can check that,

$$\{c_n, c_m^\dagger\} = \delta_{m,n}, \{c_n, c_m\} = \{c_n^\dagger, c_m^\dagger\} = 0 \tag{36}$$

$$[\sigma_n^+, \sigma_m^-] = \delta_{n,m}\sigma_n^z, [\sigma_n^z, \sigma_m^\pm] = \pm 2\delta_{n,m}\sigma_n^\pm \tag{37}$$

(Only the Pauli matrix with the same site index should consider the commutation relation $\{\sigma_i^a, \sigma_j^b\} = 2\delta_{ij}\delta_{ab}, [\sigma_i^+, \sigma_j^-] = \delta_{ij}\sigma_j^z$.)

Under the Jordan-Wigner transformation, the Hamiltonian becomes

$$\begin{aligned} H &= \sum_n \sigma_n^z - \sum_n \sigma_n^x \sigma_{n+1}^x \\ &= \sum_{n=1}^{N} (1 - 2c_n^\dagger c_n) - \sum_{n=1}^{N-1} [c_n^\dagger c_{n+1}^\dagger + c_n^\dagger c_{n+1} + h.c.] + (c_N^\dagger c_1^\dagger + c_N^\dagger c_1 + h.c.) e^{i\pi\mathcal{N}}, \mathcal{N} = \sum_n c_n^\dagger c_n \end{aligned} \tag{38}$$

with

$$\begin{aligned}\sigma_n^x \sigma_{n+1}^x &= [\prod_{m<n}(1-2c_m^\dagger c_m)](c_m + c_m^\dagger)[\prod_{k<n+1}(1-2c_k^\dagger c_k)](c_k + c_k^\dagger) \\ &= (c_n^\dagger + c_n)(1-2c_n^\dagger c_n)(c_{n+1} + c_{n+1}^\dagger) \\ &= c_n^\dagger c_{n+1} + c_n^\dagger c_{n+1}^\dagger + h.c. \end{aligned} \tag{39}$$

The boundary term comes from that $\sigma_N^x \sigma_1^x = e^{i\pi \sum_{j<L} n_j} c_N^\dagger c_1 = -e^{i\pi \sum_{j\leq L} n_j} c_N^\dagger c_l = -e^{i\pi \mathcal{N}} c_N^\dagger c_l$, because to the left of $c_N^\dagger$ we certainly have $n_N = 1$. This shows that boundary condition are changed by fermion parity $e^{i\pi\mathcal{N}} = (-1)^{\mathcal{N}}$ and periodic boundary condition become anti-periodic boundary condition when $\mathcal{N}$ is even. And odd $\mathcal{N}$ relates to periodic boundary condition. Therefore, the real spin problem is not exactly the same with free fermion. Next, for odd $\mathcal{N}$, we set $e^{ikN} = 1, k = \frac{2\pi n}{N}, n = -N/2+1, ..., 0, ..., N/2$, for even $\mathcal{N}$, we set $e^{ikN} = -1, k = \pm\frac{\pi(2n-1)}{N}, n = 1, ..., N/2$.

In terms of momentum space $c_j = \frac{1}{\sqrt{N}}\sum_k e^{ikj} c_k$, the Hamiltonian becomes

$$\begin{aligned} H &= -\sum_k [2\cos(k) c_k^\dagger c_k + (e^{ik} c_k^\dagger c_{-k}^\dagger + h.c.)] + \sum_k (2c_k^\dagger c_k - 1) \\ &= \sum_k [(1-\cos(k))(c_k^\dagger c_k - c_{-k} c_{-k}^\dagger) - (e^{ik} c_k^\dagger c_{-k}^\dagger + h.c.)] \\ &= \sum_{k>0} [(1-\cos(k))(c_k^\dagger c_k - c_{-k} c_{-k}^\dagger) - (e^{ik} c_k^\dagger c_{-k}^\dagger + h.c.)] + \sum_{k<0} ... \\ &= \sum_{k>0} [2(1-\cos(k))(c_k^\dagger c_k - c_{-k} c_{-k}^\dagger) - (2i\sin(k) c_k^\dagger c_{-k}^\dagger - 2i\sin(k) c_{-k} c_k)] \\ &= \sum_{k>0} (c_k^\dagger, c_{-k}) \begin{pmatrix} 2(1-\cos(k)) & -2i\sin(k) \\ 2i\sin(k) & 2(1-\cos(k)) \end{pmatrix} \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix} \end{aligned} \tag{40}$$

where we used $\sum_k 2\cos(k) c_k^\dagger c_k = \sum_k \cos(k)(c_k^\dagger c_k - c_{-k} c_{-k}^\dagger)$, and $\sum_k (2c_k^\dagger c_k - 1) = \sum_k (c_k^\dagger c_k - c_{-k} c_{-k}^\dagger)$.

The diagonalization is akin to Bogovliubov transformation, and all eigenvalues can be calculated:

$$\Lambda(k) = \pm 2\sqrt{(\cos(k)-1)^2 + \sin^2(k)} = \pm 2\sin\frac{k}{2} \tag{41}$$

So we obtain that, for odd N, we have $e^{ikN} = 1, k = \frac{2\pi n}{N}, n = -N/2, ..., 0, ..., N/2,$

$$H = \sum_{n=0}^{N-1} \Lambda^-(n)(\eta_n^\dagger \eta_n - \frac{1}{2}) + const. \tag{42}$$

$$\Lambda^-(n) = [(1-\cos\frac{2\pi n}{N})^2 + (\sin\frac{2\pi n}{N})^2]^{1/2} = 2\sin\frac{2\pi n}{2N} \tag{43}$$

where Bogoliubov particle as $\begin{pmatrix} c_q \\ c_{-q}^\dagger \end{pmatrix} = \begin{pmatrix} u_q & -iv_q \\ -iv_q & u_q \end{pmatrix} \begin{pmatrix} \eta_q \\ \eta_{-q}^\dagger \end{pmatrix}$. (We have used that $H = \sum_{k>0} \Lambda(k)(\eta_k^\dagger \eta_k + \eta_{-k}\eta_{-k}^\dagger) = \sum_k \Lambda(k)(\eta_k^\dagger \eta_k - 1/2)$) For even N, we know the boundary condition is $e^{ikN} = -1, k = \pm\frac{\pi(2n-1)}{N}, n = 1, ..., N/2.$

$$H = \sum_{n=1}^{N/2} \Lambda^+(n)(\eta_n^\dagger \eta_n - \frac{1}{2}) + const. \tag{44}$$

$$\Lambda^+(n) = [(1-\cos\frac{\pi(2n-1)}{N})^2 + (\sin\frac{\pi(2n-1)}{N})^2]^{1/2} = 2\sin\frac{\pi(2n-1)}{2N} \tag{45}$$

The expression for H in above allows to immediately conclude that the ground state of the Hamiltonian must be the Bogoliubov vacuum state $|0\rangle$ which annihilates the $\vec{\eta}_k|0\rangle = 0$ for all k. Thus, the ground state energy is

$$E_0^+ = -\frac{1}{2}\sum_{n=1}^{N/2} \Lambda^+(n) + const. = -csc\frac{\pi}{2N} + const \approx -\frac{2N}{\pi} - \frac{\pi}{12N} + ... \tag{46}$$

$$E_0^- = -\frac{1}{2}\sum_{n=1}^{N/2-1} \Lambda^-(n) + const. = -cot\frac{\pi}{2N} + const \approx -\frac{2N}{\pi} + \frac{\pi}{6N} + ... = E_0^+ + \frac{\pi}{4N} \tag{47}$$

Compare E_0^+ ($1/N$ term) with CFT, we have $c = 1/2$. And we used $csc(x) \approx= \frac{1}{x} + \frac{x}{6} + .., cot(x) \approx \frac{1}{x} - \frac{x}{3} + ..$

The lowest excited energy in even sector is

$$E_1^+ = \Lambda^+(1) + \Lambda^+(N/2) + E_0^+ = 4\sin\frac{\pi}{2N} + E_0^+ \approx \frac{2\pi}{N} + E_0^+ \tag{48}$$

Thus, compared with CFT, we have

$$\Delta = \bar{\Delta} = 1/2 \tag{49}$$

In the odd sector, we have

$$E_0^- = E_0^+ + \frac{\pi}{4N} \tag{50}$$

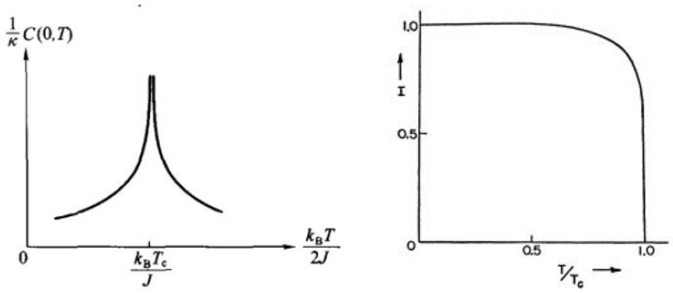


FIG. 3: Specific heat and magnetization of 2D Ising model.

Compared with CFT,

$$\Delta_\sigma = \bar{\Delta}_\sigma = 1/16 \tag{51}$$

This is related to the Majorana mode!

The same method can be applied to the case of away from the critical point, and then one can solve all energies. Once we have all energies, we can go further calculate the partition function. And all thermal quantities can be derived by the partition function. Here we omit these part of discussion and show some results directly. Around T_c, the specific heat is

$$C(T) \approx \frac{2}{\pi}(\frac{2J}{k_B T_c})^2[-\ln|1-\frac{T}{T_c}|T_c + const] \tag{52}$$

and the magnetization is [The Spontaneous Magnetization of a Two-Dimensional Ising Model, C. N. YANG, Phys. Rev. 85, 808 (1951)]

$$m(B=0,T) \approx \begin{cases} [8\sqrt{2}\frac{J}{k_B T_c}(1-\frac{T}{T_c})]^{1/8}, T \le T_c \\ 0, T > T_c \end{cases} \tag{53}$$

In Fig. 3, $C(T)$ develops a singularity at T_c. The spontaneous magnetization forms when $T < T_c$.

MEAN-FIELD SOLUTION

Weiss Mean-Field Theory

The simplest approximate treatment of spin systems in general is the mean-field theory (MFT) approximation. The general idea is to replace the many-spin system by an effective one-spin Hamiltonian in the presence of an effective external field produced collectively by the remaining spins. The approximation is valid away from the transition point, deep in the classically well-ordered and classically disordered states, where fluctuations are small.

To implement Weiss mean-field approximation on the classical spin model, we assume long-range magnetic order, characterized by a magnetization proportional to $\langle\sigma_i\rangle$, with spin then given by

$$\sigma_i = \langle\sigma_i\rangle + (\sigma_i - \langle\sigma_i\rangle) \tag{54}$$

a sum of the mean-field value and (presumed) small classical fluctuations. Inserting this into the spin Hamiltonian in the presence of an external field and neglecting the small fluctuations terms beyond first order, we obtain

$$\begin{aligned} H_{MFT} &= -J\sum_{\langle ij\rangle}[\langle\sigma_i\rangle + (\sigma_i - \langle\sigma_i\rangle)][\langle\sigma_j\rangle + (\sigma_j - \langle\sigma_j\rangle)] + B\sum_i \langle\sigma_i\rangle + (\sigma_i - \langle\sigma_i\rangle) \\ &\approx J\sum_{\langle ij\rangle}[-\langle\sigma_i\rangle\langle\sigma_j\rangle + \langle\sigma_i\rangle\sigma_j + \langle\sigma_j\rangle\sigma_i] + B\sum_i \sigma_i \\ &= \sum_i B_{eff}\cdot\sigma_i - J\sum_{\langle ij\rangle}\langle\sigma_i\rangle\langle\sigma_j\rangle \end{aligned} \tag{55}$$

where the effective Weiss field is

$$B_{eff} = B + 2J\sum_j \langle\sigma_j\rangle = B + Jz\langle\sigma\rangle \tag{56}$$

where z is the number of nearest neighbors, e.g. $z = 2$ for 1d, $z = 4$ for 2d, $z = 8$ for 3d.

It quite clearly gives a self-consistent mechanism to induce magnetic order, $\langle\sigma_i\rangle \neq 0$, even for a vanishing external magnetic field. The Weiss field on spin σ_i is generated by the neighboring spins. Since the above mean-field Hamiltonian is for a single spin, it can be solved exactly utilizing the analysis below, by including an implicit self-consistency condition through B_{eff}.

Focussing on the ferromagnetic state ($J > 0$), we take $\langle\sigma_i\rangle = m$ to be spatially uniform. Let us recall magnetization density along the applied field

$$Z_1(\beta) = e^{-\beta H} = e^{\beta\frac{Jzm^2}{2}} \sum_{s=\pm1} e^{-\beta s B_{eff}} = 2e^{\beta\frac{Jzm^2}{2}} \cosh(\beta B_{eff}) \tag{57}$$

For N noninteracting spins $Z_N = (Z_1)^N$, the free energy $F_N = NF_1$, and

$$m(B) = -\frac{\partial F_N}{\partial B} = \tanh \beta B_{eff} \tag{58}$$

Alternatively, the magnetization can also be obtained by

$$m(B) = \langle\sigma_i\rangle = Tr[\sigma_i e^{-\beta H}] = \frac{e^{\beta B_{eff}} - e^{-\beta B_{eff}}}{e^{\beta B_{eff}} + e^{-\beta B_{eff}}} = \tanh B_{eff}\beta \tag{59}$$

This is the self-consistent equation that should be satisfied by $m(B)$. One can solve it, for example by plotting (see Fig. 4). The non-trival solution depends on the slope of T, which determines the critical temperature:

$$T_c = \frac{zJ}{k_B}. \tag{60}$$

It is clear that for sufficiently high $T > T_c$, and zero external field $B = 0$, there is only a single trivial paramagnetic solution $m = 0$. However, for $T < T_c$, and there is also a nontrivial, ferromagnetic $m > 0$ solution, that can be shown to minimize the free energy for $T < T_c$. The slope of the $m \neq 0$ solution will determine the real ground state.

Here, for $T > T_c$, the system is paramagnetic, and the ground state has the symmetry with the Hamiltonian. For example, one can check the Z_2 symmetry by sending $\sigma_i \to -\sigma_i$, the Hamiltonian is unchanged. The paramagnetic state preserves this symmetry. In contrast, for $T < T_c$, $m_0 \neq 0$ and the system is ferromagnetic. Z_2 symmetry is not conserved any more, which means spin-up is not equivalent to spin-down. So the symmetry of the ground state is lower than the microscopic Hamiltonian. This phenomenon is dubbed as spontaneously symmetry breaking. We will return here in the discussion of Landau phase transition theory.

It can be verified that above expressions display the correct quantum and classical limits. In the latter, classical limit $B \ll k_B T$ the result reduces to Curie linear susceptibility

$$\chi_C(B = 0, T) = \frac{\partial m}{\partial B}|_{B\to 0} \to \frac{C}{T} \tag{61}$$

with C the Curie constant and $m \sim \chi_C(T)B$ exhibiting a linear response in this regime. This $1/T$ linear susceptibility behavior is a generic experimental signature of independent

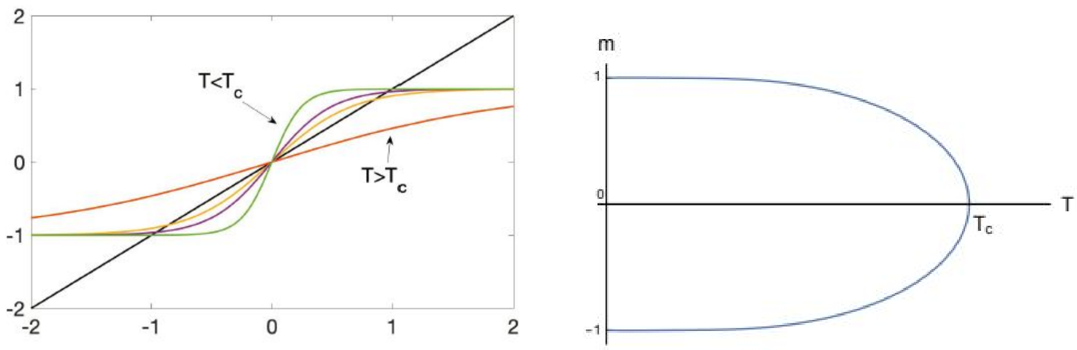


FIG. 4: Solution of mean-field self-consistent equation: $m(B) = \tanh \beta(B + Jzm)$ and the mean-field phase diagram of Ising model.

local moments, with the amplitude C a measure of the size of the magnetic moment and the associated spin. At finite T the susceptibility is finite and paramagnetic (i.e., magnetization is along the applied magnetic field and vanishes with a vanishing field), only diverging at a vanishing temperature. This captures the fact that in a classical regime, as $T \to 0$ a nonzero magnetization is induced in response to an infinitesimal field, as disordering thermal fluctuations vanish.

Critical exponents in phase transitions

Next we dive into the discussion of critical behavior around the phase transition point, and analyze the criticality.

critical behavior 1

For $T < T_c$ close to T_c , we assume magnetization is a small value $m_0 \to 0$. In this limit, we can expand $\tanh x = x - x^3/3 + ...$, and get

$$m_0 \approx m_0 Jz\beta - \frac{1}{3}(\beta Jzm_0)^3 + ... \tag{62}$$

Dividing by m_0, we can solve as

$$m_0 = \pm\sqrt{3}(\frac{k_B T}{zJ})^{3/2}(\frac{zJ}{k_B T} - 1)^{1/2} \tag{63}$$

Using $T_c = zJ/k_B$, we have

$$m_0 = \pm\sqrt{3}(\frac{T}{T_c})^{3/2}(\frac{T_c}{T} - 1)^{1/2} \tag{64}$$

Here, Mean-field theory (correctly) predicts that the magnetization vanishes as a power law with temperature. The critical exponent related to the magnetization is conventionally called $\beta = 1/2$.

In comparison, we can compare it with the Ising magnet in 3d, the magnetization vanishes as $M \sim (T_c - T)^{0.315}, T < T_c$; and in the two-dimensional Ising model, the exact computations give $M \sim (T_c - T)^{1/8}, T < T_c$.

critical behavior 2

In the ferro-magnetic phase, we consider thermal susceptibility per spin

$$\chi(T, B) = \frac{\partial m}{\partial B}|_T \tag{65}$$

We expand the self-consistent equation like

$$m = \tanh(\beta B + zJ\beta m) \approx \beta B + \frac{\beta}{\beta_c} m - \frac{1}{3}(\beta B + zJ\beta m)^3 \tag{66}$$

$$\Rightarrow \beta B \approx (1 - \frac{T_c}{T})m + \frac{1}{3}(\frac{T_c}{T})^3 m^3 \tag{67}$$

We can calculate $\partial B/\partial m$ first and then calculation the susceptibility in the case of $B = 0$:

$$\chi(T, B = 0) = [\frac{\partial B}{\partial m}]^{-1} = [kT(\frac{T_c}{T})^3 m_0^2 + kT(1 - \frac{T_c}{T})]^{-1} \tag{68}$$

The susceptibility diverges with a power law both above and below the critical temperature T_c:

$$\chi(T) = \frac{1}{A_\pm |T - T_c|^\gamma} \tag{69}$$

and the critical exponent is $\gamma = 1$.

critical behavior 3

Again, from $m = \tanh \beta(B + Jzm)$ and $k_B T_c = Jz$ near the critical T_c, we have the magnetization in the non-zero magnetic field

$$m = \tanh(m + \frac{B}{zJ}) \approx m + \frac{B}{Jz} - \frac{m^3}{3} + ... \tag{70}$$

Hence, $m \sim B^{1/3}$. This gives a critical exponent $\delta = 3$.

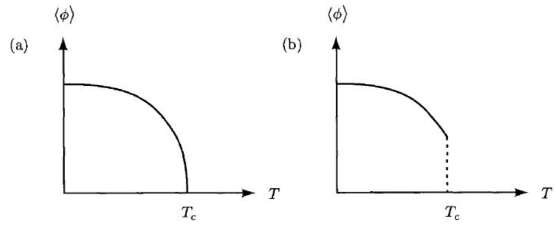


FIG. 5: Order parameter changes around the critical point.

critical behavior 4

*H*omework: Please calculate the specific heat satisfy with

$$C = \begin{cases} 0, T \to T_c^+ \\ 3NkT, T \to T_c^- \end{cases} \tag{71}$$

So there is no critical behavior in specific heat, or exponent $\alpha = 0$.

At last, several remarks are given in order. Mean-field theory correctly describes the qualitative features of most phase transitions and, in some cases. Since mean-field theory replaces the actual configurations of the local variables (e.g. spins) by their average value, it neglects the effects of fluctuations about this mean. These fluctuations may or may not be important. The more spins that interact with a particular test spin, the more the test spin sees an effective average or mean field. If the test spin interacts with two neighbors, the averaging is minimal and the fluctuations are large and important. The number of spins producing the effective field increases with the range of the interaction and with the dimension. Thus we deduce that mean-field theory is a good approximation in high dimensions but fails to provide a quantitatively correct description of second-order critical points in low dimensions.

Before proceeding, let us review some simple facts about phase transitions. At high temperatures, there is no order, and the order parameter m is zero. At a critical temperature, T_c, order sets in so that, for temperatures below T_c, m is nonzero. If m rises continuously from zero, as shown in Fig. 5, the transition is second order. If order parameter jumps discontinuously to a nonzero value just below T_c, the transition is first order.

We have seen that the thermal quantities becomes singular near the critical point (temperature). And we have defined four exponents $(\alpha, \beta, \gamma, \delta)$ related to physical quantities.

In the traditional way, there are two more critical exponents that are usually discussed. One is called ν, it appears in the correlation length in the correlation function $G(r) \sim \frac{1}{r}e^{-r/\xi}$:

$$\xi \sim |T - T_c|^{-\nu} \tag{72}$$

The mean-field value is $\nu = 1/2$. The other one is η, and it also appears in the Green's function at the critical point $G(r) \sim r^{-d+2-\eta}$. Its mean-field value is $\eta = 0$.

Traditionally, people conclude that the above six critical exponents satisfy the relationships

$$\alpha + 2\beta + \gamma = 2, Rushbrooke \tag{73}$$

$$\gamma = \beta(\delta - 1), Widom \tag{74}$$

$$\gamma = \nu(2 - \eta), Fisher \tag{75}$$

$$\nu d(2 - \alpha), Josephson \tag{76}$$

So only two independent critical exponents.

Nowadays, it has been established that the critical exponents should depend on the spatial dimension, the symmetry of the order parameter, and the symmetry and range of interactions, but not on the detailed form and magnitude of interactions. Thus, there are universality classes, and all transitions in the same universality class have the same critical exponents. For example, all transitions in which the order parameter has up-down symmetry ($n = 1$, Ising) should have the same exponents. To study these universality is the key problem in the study of phase transition.

The comparison between mean-field theory and experiments, please see Fig. 6. Critical exponents for most experimental second-order transitions differ from those predicted by mean-field theory.

Furthermore, experiments reveal that, the above critical exponets looks like universal, independent of experimental details. For some quite different systems, they may share the very similar critical behavior! (Be caution: the experimental data is actually not so good.) This drives the "unversality hypthesis": Two different systems with the same dimension d, the same symmetry may take the same critical behavior, independent of the form of interactions or other details.

Here we further show one example: The Ising universality class is characterized by the same critical exponents in the van der Waals fluids near the critical gas-liquid phase tran-

Exponent	α	β	γ	ν	η
Property	specific heat	order parameter	susceptibility	coherence length	correlation function
Definition	$C \sim t^{-\alpha}$	$\langle\phi\rangle \sim t^{\beta}$	$\chi \sim t^{-\gamma}$	$\xi \sim t^{-\nu}$	$G(q) \sim q^{-2+\eta}$
Mean-field	0	0.5	1	0.5	0
3D theory					
$n=0$ (SAW)	0.24	0.30	1.16	0.59	
$n=1$ (Ising)	0.11	0.32	1.24	0.63	0.04
$n=2$ (*xy*)	−0.01	0.35	1.32	0.67	0.04
$n=3$ (Heisenberg)	−0.12	0.36	1.39	0.71	0.04
Experiment					
3D $n=1$	$0.11^{+.01}_{-.03}$	$0.32^{+.16}_{-.04}$	$1.24^{+.16}_{-.04}$	$0.63^{+.04}_{-.04}$	0.03 - 0.06
3D $n=3$	$0.1^{+.05}_{-.04}$	$0.34^{+.04}_{-.04}$	$1.4^{+.07}_{-.07}$	$0.7^{+.03}_{-.03}$	
2D $n=1$	$0.0^{+.01}_{-.003}$	$0.3^{+.04}_{-.04}$	$1.82^{+0.7}_{-.07}$	$1.02^{+.07}_{-.07}$	

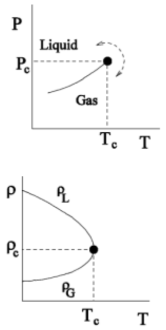
Experiments on 3D $n=1$ compiled from liquid-gas, binary fluid, ferromagnetic, and antiferromagnetic transitions.
Experiments on 3D $n=3$ transitions compiled from some ferromagnetic and antiferromagnetic transitions.
Experiments on 2D $n=1$ complied from some antiferromagnetic transitions.

FIG. 6: Some critical exponents from theory and experiments (Copyright from *Principles of condensed matter physics*, Cambridge University Press).

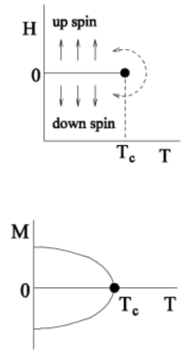
sition. It is summarized below. The liquid-gas transition has much in common with the

Mean field Ising universality class

Correspondence between magnetics and fluids



Gas-Liquid	Magnets
Volume, V or density, ρ	Mean magnetization, $-M$
Pressure, P	Magnetic field, B
Gibbs free energy, $G(P,T)$	Gibbs free energy, $G(B,T)$
Compressibility, $\kappa_T = -\frac{1}{V}\frac{\partial V}{\partial P}$	Susceptibility, $\chi = \frac{\partial M}{\partial B}$
Heat capacity, $C_P = -T\left(\frac{\partial^2 G}{\partial T^2}\right)_P$	Heat capacity, $C_B = -T\left(\frac{\partial^2 G}{\partial T^2}\right)_B$



Density (Order parameter) $\rho(T,P_c) \sim (T_c - T)^{\beta}$, $\beta_{MF} = \frac{1}{2}$
Critical isotherm $V(P,T_c) \sim P^{\delta}$, $\delta_{MF} = 3$
Compressibility $\kappa_T(T) \sim |T_c - T|^{-\gamma}$, $\gamma_{MF} = 1$
Heat capacity $C_P(T) \sim |T_c - T|^{-\alpha}$, $\alpha_{MF} = 0$

FIG. 7: Liquid-gas transition v.s. PM-FM transition.

magnetic transition in an Ising model. We compare the phase diagram in the $h - T$ plane for the Ising model to the phase diagrams in the $p - T$ planes for a fluid near its critical point. In both the Ising model and the fluid, there is a coexistence curve, terminating at a critical point, along which two distinct but equal free energy phases coexist, and in both it is possible to go continuously around the critical point from one coexisting phase to the other by appropriately varying h and T. The coexistence curve for the magnet is a straight line $h = 0$, $T < T_c$, whereas that for the fluid is in general curved. The inversion ($m \to -m$) symmetry of a magnet forces the coexistence line to be the line $h = 0$ and the critical point value of h and m both to be zero. There is no such symmetry in a fluid and no special values of the critical point parameters p_c, T_c.

LANDAU PHASE TRANSITION THEORY

Here we begin to discuss the Landau's continuous phase transition theory. There are two very important concepts: Order parameter and symmetry spontaneously broken, which are the cornerstone of condensed matter physics.

Let us recall the implicit self-consistent MFT Eq. 58 for $m(B, T)$ is actually a saddle-point equation for the free- energy density $f(m, B, T)$ with respect to m, i.e., corresponds to $\partial f/\partial m = 0$. As we are interested in the behavior near the critical point, where the magnetization is small, $m \ll 1$. Thus, Eq. 58 lead to a free energy that is quartic polynomial in the magnetization m, with a quadratic coefficient and quartic one a positive constant.

While above mean-field analysis relies on a specific microscopic model, as was first argued by Lev D. Landau (1937), above mean-field predictions are much more universal and are a consequence of continuous phase transition. Guided by general symmetry principles, Landau postulated a phenomengical theory, i.e. near a continuous phase transition the free energy density exhibits a generic analytic expansion in powers of an order parameter, the magnetization, in the case of a PM-FM transition (see free energy of Eq. 57),

$$\begin{aligned} f &= -k_BT\ln Z_1 = \frac{Jzm^2}{2} - k_BT\ln 2 - k_BT\ln[\cosh(\beta B_{eff})] \\ &= \frac{Jzm^2}{2} - k_BT\ln 2 - k_BT[\frac{(\beta B_{eff})^2}{2} - \frac{(\beta B_{eff})^4}{12} + ...] \\ &\overset{B_{eff}=mzJ}{=} \frac{Jzm^2}{2} - k_BT\ln 2 - k_BT[\frac{(\beta mzJ)^2}{2} - \frac{(\beta mzJ)^4}{12} + ...] \\ \Rightarrow f &= f_0 + a(T)\mathbf{m}^2 + b(T)\mathbf{m}^4 \end{aligned} \tag{77}$$

where $a(T) = \frac{k_BT_c}{2T}(T - T_c) = a_0(T - T_c)$ and $b = \frac{k_BT_c^4}{12T^3}$.

The form is dictated by the time-reversal symmetry of the Hamiltonian for B = 0 (for Ising case, $m \to m$ is a symmetry for B = 0, dictating that no odd powers of m appear in f), with coefficients smooth functions of T, and, crucially $a(T) = a_0(T/T_c - 1)$, changing sign to $a(T < T_c) < 0$. In the presence of magnetic field, one can add a term hm to the free energy, which break the even parity of m.

In the Ising case for $a(T > T_c) > 0$, $f(m)$ is well-approximated by a parabola, with a single minimum at the origin, $m = 0$ (see Fig. 8). In contrast, for $a(T < T_c) < 0$, the free energy develops a symmetric double-well form, minimized by a finite magnetization, $m_0 = a/b \sim |T - T_c|^{1/2}$. Thus, this generic Landau theory indeed predicts the phenomenology

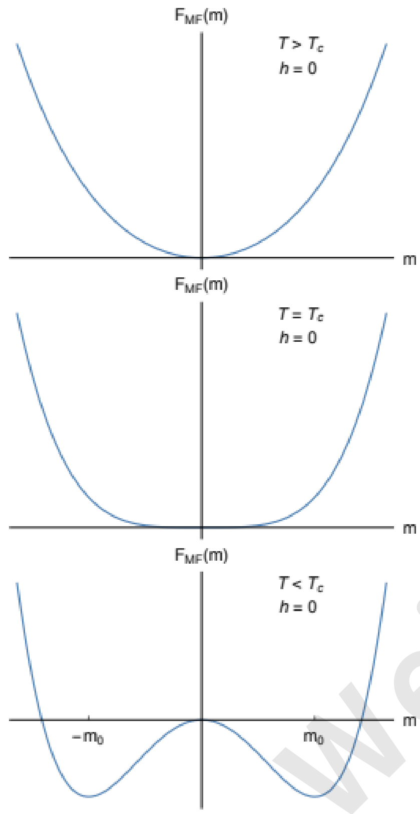


FIG. 8: Mean field solution of the free energy as a function of magnetization at zero field at various temperatures. As we cool the system below T_c, the free energy changes smoothly to a symmetry breaking case, thereby making the $m = 0$ solution unstable and two new stable solutions at $m = \pm m_0$ appear.

near T_c found above.

In the discussion of susceptibility, we can change the form of free energy to

$$f = f_0 + a(T)m^2 + b(T)m^4 - hm + ... \tag{78}$$

The configurations of minimum F are spatially uniform and so require that

$$\frac{\partial f}{\partial m} = 0 \tag{79}$$

yields $2am + 4bm^3 = h$. This again relates the coulpings in the Landau theory. When $h = 0$, we have $m = \pm\sqrt{\frac{-a}{2b}}$. If we assume b is constant, we have $a = \frac{T-T_c}{T_c}$ when $T < T_c$ to satisfy Eq. 64. The susceptibility is obtained by $\chi(B,T) = \frac{\partial m}{\partial B} = \frac{1}{2a+8bm^2}$, so the $\chi(B = 0,T) \sim 1/|T - T_c|$ can be obtained. Also, at $T = T_c$, we have $h = 4bm^3$, which is consistent with $m \sim h^{1/3}$. So the Landau mean-field theory is totally consistent. *Landau*

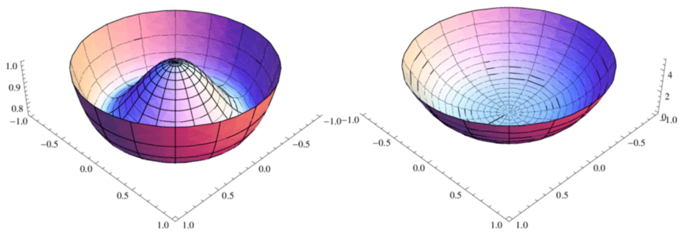


FIG. 9: Left is potential of free energy in symmetry breaking phase and the right is that in symmetric phase. In the symmetry breaking phase, the low-energy excitation is gapless Goldstone mode.

theory is remarkable in that, under the simple assumptions that the order parameter is small and uniform near Tc, it yields a wealth of information about phase transitions.

A new crucial ingredient will arise for the case of a multi-component vector order-parameter, $\vec{m}$. While MFT exponents remain the same, the Landau free-energy potential exhibits zero-energy (the so-called) Goldstone modes, corresponding to reorientation of the order parameter, that is, the motion along the minimum of the "Mexican hat" potential. This case is different from the discrete Z_2 order in Ising.

Order parameters and symmetry spontaneously broken

With all above preparation, here we give some formal definitions:

Order parameter.— It is a parameter to indicate a phase transition. It usually relates to some physical observable. Example is magnetization m in the Ising transition.

Symmetry spontaneously breaking.— Landau's theory on phase transition is based on transition from HIGH symmetry state to a LOW symmetry state. This symmetry reducing process is called symmetry spontaneously breaking. In the above PM-FM transition, PM phase preserves the Z_2 transition while FM phase breaks this symmetry. So the Ising transition is a Z_2 transition.

Goldstone mode.— There are two symmetry breaking transition. One is discrete symmetry breaking, as PM-FM transition. The other type is continuous symmetry breaking. The free energy changes is like Fig. 9. In the symmetric breaking phase, the low-energy excitation is gapless, which is called Goldstone mode. Examples include phonon, magnon, etc.

Here we would like to give some more examples of phase transitions.

- Heisenberg Ferromagnet.

  Order Parameter in the Heisenberg ferromagnet (in, say, 3D), the spin can point anywhere in space. Hence the magnetization is a vector, defined in the same was as for the Ising Model:

$$\mathbf{m_i} = \langle \mathbf{S}_i \rangle \tag{80}$$

  The free energy becomes:

$$f = f_0 + a(T)\mathbf{m}^2 + b(T)\mathbf{m}^4 - h \cdot \mathbf{m} + ... \tag{81}$$

  Any free energy must be invariant under a rotation

$$\mathbf{m_i} \to \mathbf{R} \cdot \mathbf{m_i} \tag{82}$$

  In this case, rotation is a continuous symmetry (i.e. SO(3) in 3D). This is the primary difference between Ising and Heisenberg: the Ising magnet breaks up-down symmetry, while the Heisenberg magnet breaks rotational symmetry: M can point anywhere in space.

- Ferroelectricity.

  Order Parameter in polarization or electric dipole moment, which is from the inversion symmetry spontaneously breaking. Around the phase transition temperature, the crystal structure should change (the location of ions changes).

  Material BaTiO3 is shown in Fig. 10. Above T_c it is P_m3_m while below T_c it is P_4mm. Around the phase transition, Ti and O move along $+Z$ $-Z$ direction.

More on discrete symmetry

The simplest discrete group is the group Z_2 consisting only of two elements: the identity and an element whose square is the identity $\{1, -1\}$. Realizations of this group include the group of reflections about a plane, time reversal, and so on. Z_2 symmetry is broken in any

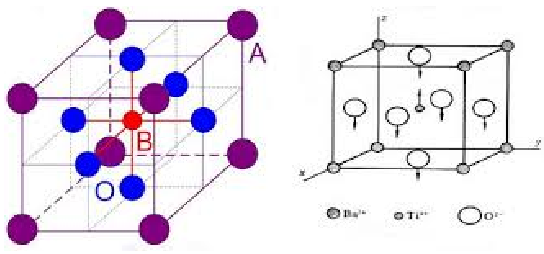


FIG. 10: BTO crsytal structure changes at T_c, resulting in a spontaneous polarization.

phase transition in which there are two and only two equivalent ordered states characterized by order parameters that can be chosen to differ only in sign. Though Z_2 symmetry is the most important and most common discrete symmetry encountered in condensed matter physics, there are, of course, many others. A particularly useful hierarchy of discrete groups for studying properties of discrete symmetry is the set of groups Z_N of integers under addition modulo N.

A class of models with Z_N symmetry, called clock models, can be defined by associating with each site on a lattice a spin variable $s_i = (\cos(2\pi n_i)/N, \sin(2\pi n_i/N))$ of unit length:

$$H = -J \sum_{n_i, n_j} \cos(2\pi(n_i - n_j)/N) \tag{83}$$

where $n_i = 0, ..., N-1$.

More on continuous symmetry

The simplest continuous group is the two-dimensional orthogonal group O_2 of rotations in a two-dimensional plane. Since a two-dimensional vector is equivalent to a complex number, the group O_2 is isomorphic to the group $U(1)$ of transformations of the phase of a complex number. The symmetry associated with these groups is often called xy-symmetry because rotations are usually done in the xy -plane. The group 0_2 is of enormous pedagogical important in the discussion of KT-transition. The simplest realization of a system with O_2 symmetry is an easy-plane ferromagnet in which spins are confined by crystal fields to lie in the xy-plane. The $O(2)$or xy-model can be reexpressed in terms of a local angle variable by setting $s_i = (\cos\theta_i, \sin\theta_i)$:

$$H = -J \int d\theta_i \theta_j \cos(\theta_i - \theta_j) \tag{84}$$

where θ_i is a continuous variable. Please be aware the different in comparison with Z_N clock model.

The most important phase with broken $U(1)$ symmetry is superfluid helium. The superfluid phase is characterized by a macroscopic condensate wave function with an amplitude and a phase.

The group 0_3 of rotations in three dimensions is another continuous group of considerable importance. Physical realizations of systems with this symmetry include Heisenberg ferromagnets and antiferromagnets on lattices where crystal fields aligning spins along crystal axes are unimportant.

FLUCTUATING FIELD AND ϕ^4 THEORY

The long-range character of critical fluctuations and the universality of the critical properties suggest that it is possible to calculate the critical behavior by a phenomenological field theory rather than a microscopic model. One may neglect the lattice completely and describe the partition function of the system near T_c by a functional integral over a continuous local order field $\phi(x)$. The energy functional is assumed to have a Taylor expansion in the order field $\phi(x)$ and in its gradients. Here we derive this phenomenological theory, which will appear in our future study.

Let us start from the partition function

$$\begin{aligned} Z &= \sum_{\sigma_1=\pm,\ldots,\sigma_N=\pm} \exp[-\beta \sum_{\langle ij \rangle} \sigma_i \sigma_j] = \sum_{\{\sigma_i\}} \exp[-\sum_{\langle ij \rangle} \sigma_i K_{ij} \sigma_j] \\ &= const. \int d\mathbf{v} \sum_{\{\sigma_i\}} e^{-\frac{1}{2}\sum_{ij} v_i (K_{ij}^{-1}) v_j + \sum_i \sigma_i v_i} \\ &= const. \int D v e^{-\frac{1}{2}\sum_{ij} v_i (K_{ij}^{-1}) v_j} \prod_i (2\cosh(v_i)) \\ &\overset{\phi_i = \sum_j K_{ij}^{-1} v_j}{=} const. \int D\phi e^{-\sum_{ij} \phi_i K_{ij} \phi_j + \sum_i \ln\cosh(2\sum_j K_{ij}\phi_j)} \end{aligned} \tag{85}$$

where we introduce a auxiliar field v_i and the Gaussian integral as

$$\int d\mathbf{v} e^{-\frac{1}{2}\vec{v}\cdot A \cdot \vec{v} + \vec{v}\cdot\vec{j}} = (2\pi)^{N/2}[detA]^{-1/2} e^{\frac{1}{2}\vec{v}\cdot\vec{j}} \tag{86}$$

Then we assume the field is small $|\phi(x)| \ll 1$, and the spatial profile of the field is smooth, we can expand $\ln\cosh(x) = x^2/2 - x^4/12 + ...$ If writing in the Fourier space,

$\phi_i = \frac{1}{N}\sum_k e^{i\mathbf{k}\cdot\mathbf{r}_i}\phi(\mathbf{k})$ and $K_{ij} = \frac{1}{N}\sum_k e^{i\mathbf{k}\cdot(\mathbf{r}_i-\mathbf{r}_j)}K(\mathbf{k})$, we obtain that

$$S[\phi] = \sum_k \phi_{-\mathbf{k}}[c_0 + c_2 k^2]\phi_{\mathbf{k}} + c_4 \sum_{k_1,k_2,k_3,k_4} \delta_{\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3}+\mathbf{k_4}=0}\phi_{\mathbf{k}_1}\phi_{\mathbf{k}_2}\phi_{\mathbf{k}_3}\phi_{\mathbf{k}_4} + ... \tag{87}$$

where we used $K(k) = K(0) + \frac{1}{2}k^2 K''(0) + ...$ If write in the spatial space, we have

$$S[\phi] = \int d^d x (c_2(\partial\phi)^2 + c_0\phi^2 + c_4\phi^4) \tag{88}$$

This is the so-called ϕ-4 theory or action. It is a phenomenogical theory of Ising model.

Also, this functioanl is also called the Ginzburg-Landau functional. It differs from Landaus expansion of the Gibbs free energy Eq. 57 by the first gradient term (apart from the more general notation).