Fractional Statistics

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The necessity of quantum statistics is motivated by the indistinguishability of identical particles. In dimensions $d = 3$ (and possibly higher), the fact that the quantum state $|\Psi\rangle$, say, of two identical, non-interacting particles should be invariant with respect to their adiabatic exchange, leads to the representation of $|\Psi\rangle$ either as a symmetric, or an antisymmetric superposition of the quantum states of each particle, i.e.

$$\Psi(\mathbf{r}_1, \mathbf{r}_2) = \pm\Psi(\mathbf{r}_2, \mathbf{r}_1) \tag{1}$$

Correspondingly, particles can be classified as either bosons or fermions, respectively, depending on whether they obey the Fermi-Dirac or the Bose-Einstein quantum statistics. For fermions, everyone knows the Pauli exclusion principle. The spin-statistics theorem then fixes the spin of fermions and bosons as a semi-odd-integer or integer multiple of $\hbar$, respectively.

Such a restriction, however, can be overcome in $d = 2$. In particular, in two dimension any statistics would be possible, in principle, with properties somehow interpolating between those of Fermi-Dirac and of Bose-Einstein statistics. In particular, the fact that $\Psi(r_1, r_2) = e^{i\alpha\pi}|\Psi(r_1, r_2)\rangle$ implies in general a global phase $\alpha\pi$, not necessarily equal to 0 or π, as is the case for bosons and fermions, respectively. Since this phase is expressed as a fraction α of the phase π for fermions, one also speaks of fractional exchange statistics. In this context, the number α, ranging from $\alpha = 0$ (bosons) to $\alpha = 1$ (fermions), is also referred to as the statistical parameter. As mentioned above, fractional exchange statistics is usually restricted to two spatial dimensions ($d = 2$). Interestingly, fractional exchange statistics can be formalized, to some extent, also in $d = 1$.

Current interests on fractional statistics are mainly motivated by its relevance for fractional quantum Hall effects, and quantum spin liquids. And very recently, it is widely studied in the topological order. I believe this part is the most exciting field in physics nowadays.

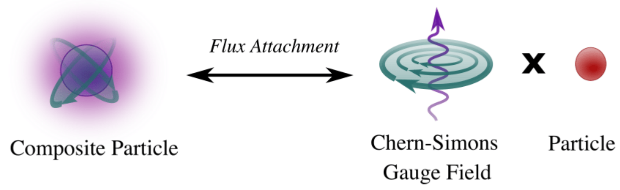

FIG. 1: Flux attachment scheme: A composite particle is made of a bare particle with a flux tube.

FLUX ATTACHMENT SCHEME

We consider a Hamiltonian for N-fermion:

$$H = \sum_{i=1}^{N} \frac{1}{2m}[\mathbf{p}_i^2 - e\mathbf{A}(\mathbf{r}_i)]^2 + \sum_i eA_0(\mathbf{r}_i) + \sum_{i<j} V(\mathbf{r}_i - \mathbf{r}_j) \tag{2}$$

$\mathbf{A}(\mathbf{r}_i)$ and A_0 are the vector potentials. Because we are dealing with fermions, the wave function in the eigenvalue problem,

$$H\Psi(\mathbf{r}_1, ...\mathbf{r}_N) = E\Psi(\mathbf{r}_1, ..., \mathbf{r}_N) \tag{3}$$

$$\Psi(\mathbf{r}_1, ...\mathbf{r}_i, ..., \mathbf{r}_j, ...\mathbf{r}_N) = -\Psi(\mathbf{r}_1, ...\mathbf{r}_j, ..., \mathbf{r}_i, ...\mathbf{r}_N) \tag{4}$$

Next, we try to consider a different problem, by introducing a new vector potential (gauge field) $\mathcal{A}(\mathbf{r})$ to each electron:

$$H' = \sum_{i=1}^{N} \frac{1}{2m}[\mathbf{p}_i^2 - e\mathbf{A}(\mathbf{r}_i) + e\mathcal{A}(\mathbf{r}_i)]^2 + \sum_i eA_0(\mathbf{r}_i) + \sum_{i<j} V(\mathbf{r}_i - \mathbf{r}_j) \tag{5}$$

and the new vector potential ($\phi_0 = 2\pi/e$)

$$\mathcal{A}(\mathbf{r}_i) = \frac{\phi_0}{2\pi}\frac{\theta}{\pi}\sum_{j\neq i} \nabla_i \alpha_{ij} \tag{6}$$

where $\alpha_{ij} = \arctan\frac{y_j - y_i}{x_i - x_i}$ is the angle of $\mathbf{r}_i - \mathbf{r}_j$.

Physically, this picture can be viewed as shown in Fig. 1. ...

The gauge field induced by each flux. Let us define two electrons at position $\mathbf{r}_i, \mathbf{r}_j$, respectively. α_{ij} is the angle between the x axis and the difference vector between $\mathbf{r}_j$ and $\mathbf{r}_i$ (i.e., $\mathbf{r}_j - \mathbf{r}_i$). Here, the value α_{ij} is determined only up to 2π. The angle can be expressed by the coordinates as:

$$\alpha_{ij} = \arctan\frac{y_j - y_i}{x_i - x_i} \tag{7}$$

Next, we consider one electron is attached by a flux θ (in unit of π), and ask what the vector potential is in the real space. So the problem is to solve the vector potential $\mathcal{A}(\mathbf{r})$ for the condition

$$\nabla \times \mathcal{A}(\mathbf{r}) = \phi_0 \frac{\theta}{\pi} \delta(\mathbf{r}) \hat{e}_z \tag{8}$$

One can check that, the solution is

$$\mathcal{A}(\mathbf{r}) = \frac{\phi_0}{2\pi} \frac{\theta}{\pi} \frac{1}{r} \hat{e}_\varphi = \frac{\phi_0}{2\pi} \frac{\theta}{\pi} (\frac{-y}{|\mathbf{r}|^2}, \frac{x}{|\mathbf{r}|^2}) \tag{9}$$

since

$$\oint_C \mathbf{A}(\mathbf{r}) \cdot d\mathbf{r} = \frac{\phi_0}{2\pi} \frac{\theta}{\pi} \int_0^{2\pi} r d\varphi \frac{1}{r} = \phi_0 \frac{\theta}{\pi} \tag{10}$$

Importantly, one may notice that, the vector field, that is generated by electron at position $\mathbf{r}_j$ is experienced by the other electron at position $\mathbf{r}_i$, can be expressed by

$$\mathcal{A}(\mathbf{r}_i, \mathbf{r}_j) = \frac{\phi_0}{2\pi} \frac{\theta}{\pi} (-\frac{y_i - y_j}{|\mathbf{r}_i - \mathbf{r}_j|^2}, \frac{x_i - x_j}{|\mathbf{r}_i - \mathbf{r}_j|^2}) = \frac{\phi_0}{2\pi} \frac{\theta}{\pi} \nabla_i \alpha_{ij} \tag{11}$$

where the angle α_{ij} has been defined above.

Suppose that we have N electrons in the system, the total vector potential generated by $N-1$ electrons are:

$$\mathcal{A}(\mathbf{r}_i) = \sum_{j \neq i} \mathcal{A}(\mathbf{r}_i, \mathbf{r}_j) = \frac{\phi_0}{2\pi} \frac{\theta}{\pi} \sum_{j \neq i} \nabla_i \alpha_{ij} \tag{12}$$

Next we can write the vector potential $\mathcal{A}$ into the continntum form as

$$\mathcal{A}(\mathbf{r}_i) = \frac{\phi_0}{2\pi} \frac{\theta}{\pi} \int d\mathbf{r}' \frac{\partial}{\partial x_i} \alpha(\mathbf{r} - \mathbf{r}') \rho(\mathbf{r}') \tag{13}$$

where $\rho(r)$ is charge density. Then we observe that this vector field is a pure gauge, i.e. $\mathcal{A}_\mu(\mathbf{r}) = \frac{\partial}{\partial x_\mu} \Lambda(\mathbf{r})$ where $\Lambda(\mathbf{r}) = \frac{\phi_0}{2\pi} \frac{\theta}{\pi} \alpha(\mathbf{r} - \mathbf{r}') \rho(\mathbf{r}')$.

$\mathcal{A}$ can be removed through the gauge transformation

$$\mathcal{A}_\mu \to A'_\mu = A_\mu - \partial_\mu \Lambda = 0 \tag{14}$$

involves correspondingly a gauge redefinition of the field operators as

$$\Psi(x) \to \Psi'(x) = e^{i\Lambda(x)}\Psi(x) = e^{i\frac{\phi_0}{2\pi}\frac{\theta}{\pi}\alpha(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}')}\Psi(x) \tag{15}$$

while the Hamiltonian changes with $D_\mu \to \partial_\mu$:

$$H' = \sum_{i=1}^{N} \frac{1}{2m}[\mathbf{p}_i^2 - e\mathbf{A}(\mathbf{r}_i)]^2 + \sum_i eA_0(\mathbf{r}_i) + \sum_{i<j} V(\mathbf{r}_i - \mathbf{r}_j) \tag{16}$$

$$H'\Psi' = E'\Psi' \tag{17}$$

However, the new free fields do not obey standard anticommutation relations any longer. These are in fact replaced e.g. by the braiding relation

$$\Psi'(..., \mathbf{r}_i, ..., \mathbf{r}_j, ...) = e^{i\frac{\phi_0}{2\pi}\frac{\theta}{\pi}[\alpha(\mathbf{r}_i-\mathbf{r}_j)-\alpha(\mathbf{r}_j-\mathbf{r}_i)]}\Psi'(..., \mathbf{r}_j, ..., \mathbf{r}_i, ...) \tag{18}$$

Introducing a cut along the negative x axis and measuring angles from the positive x axis, so that all angles belong to $[\pi, \pi)$ and becomes single-valued, one has

$$\alpha(\mathbf{r}_i - \mathbf{r}_j) - \alpha(\mathbf{r}_j - \mathbf{r}_i) = \pm\pi \tag{19}$$

So, we reach that

$$\Psi'(..., \mathbf{r}_i, ..., \mathbf{r}_j, ...) = e^{i\alpha\pi}\Psi'(..., \mathbf{r}_j, ..., \mathbf{r}_i, ...) \tag{20}$$

Therefore, the gauge transformation, Eq. 14, is equivalent to transmuting fermions into anyons with statistical parameter α. Or, we have shown that this system composed of fermions coupled to attached flux fields does indeed describe anyons with an appropriate statistical parameter.

§ `What is the meaning of gauge field?`

Chern-Simons field. The above analysis is actually equivalent to the description of Chern-Simons field theory. Generally, we can write down a minimal model of electron field coupled with a Chern-Simons field as

$$S = \int d\mathbf{r}[\Psi^\dagger iD_0\Psi + \frac{1}{2m}\Psi^\dagger(D_1^2 + D_2^2)\psi + \frac{\theta}{2}\epsilon^{\mu\nu\lambda}\mathcal{A}_\mu\partial_\nu\mathcal{A}_\lambda] \tag{21}$$

where $D_\mu = \partial_\mu - ie\mathcal{A}_\mu$ is the (gauge) covariant derivative. Varying S with respect to $\mathcal{A}_\mu$, one obtains the analog of the Maxwell equations for the gauge field,

$$\epsilon^{\mu\nu\lambda}\partial_\mu A_\lambda = \frac{e}{\theta} j^\mu \tag{22}$$

where

$$\Psi^\dagger \Psi = \rho \tag{23}$$

$$j^\mu = \frac{i}{2m}(\Psi^\dagger D^\mu \phi \Psi - (D^\mu \Psi)^\dagger \Psi) \tag{24}$$

The field potential can be obtained by solving the above pseudo-Maxwell equations, for given source currents j^μ, e.g. The zeroth component yields

$$\partial_1 \mathcal{A}_2 - \partial_2 \mathcal{A}_1 = e/\theta\rho \tag{25}$$

Additionally, after the gauge transformation, we will obtain the action

$$S = \int d\mathbf{r}[(\Psi')^\dagger i\partial_0 \Psi' + \frac{1}{2m}(\Psi')^\dagger(\partial_1^2 + \partial_2^2)\Psi'] \tag{26}$$

where Ψ' satisfies the fractional statistics as discussed in Eq. 20.

Chern-Simons

LAUGHLIN'S THEORY

Landau level in symmetric gauge

$$H = \frac{1}{2m}(\mathbf{p} - e\mathbf{A})^2 \tag{27}$$

where A(r) is the vector potential that generates the magnetic field

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{28}$$

When we choose the symmetric gauge: $\mathbf{A} = B(-y/2, x/2, 0)$,

$$H = \frac{1}{2}[(-i\partial_x - \frac{y}{2})^2 + (-i\partial_y + \frac{x}{2})^2] = \frac{1}{2}[-4\frac{\partial^2}{\partial z \partial \overline{z}} + \frac{1}{4}z\overline{z} - z\frac{\partial}{\partial z} + \overline{z}\frac{\partial}{\partial \overline{z}}] \tag{29}$$

where we define $z = x - iy = re^{-i\theta}, \overline{z} = x + iy = re^{i\theta}$, $\partial_x = \partial_z + \partial_{\bar{z}}, \partial_y = -i(\partial_z - \partial_{\bar{z}})$. The ladder operators can be defined as

$$a^\dagger = \tfrac{1}{\sqrt{2}}(\tfrac{\bar{z}}{2} - 2\tfrac{\partial}{\partial z}) \tag{30}$$

$$a = \tfrac{1}{\sqrt{2}}(\tfrac{z}{2} + 2\tfrac{\partial}{\partial \bar{z}}) \tag{31}$$

$$b^\dagger = \tfrac{1}{\sqrt{2}}(\tfrac{z}{2} - 2\tfrac{\partial}{\partial \bar{z}}) \tag{32}$$

$$b = \tfrac{1}{\sqrt{2}}(\tfrac{\bar{z}}{2} + 2\tfrac{\partial}{\partial z}) \tag{33}$$

$$[a, a^\dagger] = 1, \quad [b, b^\dagger] = 1 \tag{34}$$

The hamiltonian becomes

$$H = a^+a + 1/2 \tag{35}$$

In addition, The z component of the angular momentum operator is defined as

$$L_z = -i\hbar\frac{\partial}{\partial\theta} = -\hbar(z\frac{\partial}{\partial z} - \overline{z}\frac{\partial}{\partial\overline{z}}) = -\hbar(b^\dagger b - a^\dagger a) \tag{36}$$

Exploiting the property $[H, L_z] = 0$, the eigenfunctions are chosen to diagonalize H and L simultaneously. The eigenvalue of L is denoted by $m\hbar$; with this definition the quantum number m takes values $-n, -n+1, ...$

The ground state wave function is solved by $a|0,0\rangle = 0, b|0,0\rangle = 0$. We obtain

$$\langle r|0,0\rangle = \frac{1}{\sqrt{2\pi}\ell}e^{-\frac{|z|^2}{4}}. \tag{37}$$

The other wavefunctions are obtained by (The wave function in the lowest Landau level)

$$\langle r|n,m\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}\frac{(b^\dagger)^m}{\sqrt{m!}}\langle r|0,0\rangle \tag{38}$$

Especially, the single particle states are especially simple in the lowest Landau level (n = 0):

$$\langle r|n=0,m\rangle = \frac{(b^\dagger)^m}{\sqrt{m!}}\langle r|0,0\rangle = \frac{1}{\sqrt{2\pi 2^m m!}\ell}z^m e^{-|z|^2/4} \tag{39}$$

This wave function represents an electron localized circularly in disk. The maximum of the existence probability is on the circumference of a circle of radius $\sqrt{2m}\ell$, and the spread of the wave function in the radial direction is of the order of ℓ. The expectation value of r^2 is $\langle 0,m|r^2|0,m\rangle = 2(m+1)\ell^2$. Thus the largest value of m for which the state falls inside the disk is given by $m_{max} = \pi R^2/2\pi\ell^2$,

Exercises. Prove the normalization condition of electron wave functions.

$$\begin{aligned}
\langle 0,m|0,m'\rangle &= \int \overline{\frac{1}{\sqrt{2\pi 2^m m!}\ell}z^m e^{-|z|^2/4}}\frac{1}{\sqrt{2\pi 2^{m'} m'!}\ell}z^{m'} e^{-|z|^2/4} && (40)\\
&= \frac{1}{\sqrt{2\pi 2^m m!}\ell}\frac{1}{\sqrt{2\pi 2^{m'} m'!}\ell}\int_{\psi=0}^{2\pi}\int_0^\infty drd\psi e^{-i(m-m')\psi} r^{m+m'+1} e^{-r^2} && (41)\\
&= \frac{\delta_{m,m'}}{m!}m! = \delta_{m,m'} && (42)
\end{aligned}$$

Exercises. Prove the average area of each Landau orbital.

Two-electron problem

Unsymmetrised two-particle basis states from the lowest Landau level have the form

$$\psi(z_1,z_2) \sim z_1^{l_1} z_2^{l_2} e^{-(|z_1|^2+|z_2|^2)/4} \tag{43}$$

with l non-negative integers. We will consider combinations of these that are eigenfunctions of relative and centre-of-mass angular momentum. They have the form

$$\psi(z_1,z_2) \sim (z_1-z_2)^l (z_1+z_2)^m e^{-(|z_1|^2+|z_2|^2)/4} \tag{44}$$

Laughlin wave function

When the FQHE was discovered, R. Laughlin realized that one could write down a many-body variational wave function at filling factor $\nu = 1/q$. This seminal idea opens a door the answer to the FQHE.

The many-body problem is

$$H = \sum_j [\frac{1}{2m}| - i\hbar\partial_j - e\mathbf{A}_j|^2] + \sum_{j<i} \frac{e^2}{|z_j - z_i|} \tag{45}$$

Laughlin proposed the wave function of ground state of the above hamiltonian at $\nu = 1/q$ is [R. Laughlin, Anomalous Quantum Hall Effect: An Incompressible Quantum Fluid with Fractionally Charged Excitations Phys. Rev. Lett. 50, 1395 (1983) ; ABSTRACT: This Letter presents variational ground-state and excited-state wave functions which describe the condensation of a two-dimensional electron gas into a new state of matter.]:

$$\psi_q = \prod_{j,i=1 i<j}^{N_e} (z_j - z_i)^q \prod_{j=1}^{N_e} e^{-|z_j|^2/4\ell^2} \tag{46}$$

Next we try to understand this famous wave function:

- For q being an odd integer, this wave function obeys Fermi statistics.
- The value q = 1 describes a full Landau level. Higher values ensure that the probability density falls faster to zero as a pair of particles approach each other.
- The angular momentum is $M = N_e(N_e - 1)q/2$.
- In the polynomial part of Laughlin's wave function, an electron coordinate z_i has $M = (N_e - 1)q$ as the maximum power. This is the maximum angular momentum that the electron can have, and in this state the area that this electron encloses becomes maximum. The maximum area is determined by the largest momentum: $A = 2(M + 1)\pi\ell^2$, thus the filling factor $\nu = N_e 2\pi\ell^2/A = N_e/(M + 1) = N_e/[(N_e - 1)q] \sim 1/q$.

We emphasize once again that Laughlins wave function is not based on a mathematical derivation, although we will see below that there exist some mathematical models for which it describes the exact ground state, but it is more appropriately characterised as a variational wave function.

N_e	N_H	$\langle \Psi_0 \vert \Psi_3 \rangle$
4	18	0.99804
5	73	0.99906
6	338	0.99644
7	1656	0.99636
8	8512	0.99540
9	45207	0.99406

FIG. 2: Overlap of the wave functions for a system at 1/3. Ψ_0 is the exact ground state of the Coulomb interaction.

Laughlin wave function builds in good correlations among the electrons because each electron sees an $q-$fold zero at he positions of all the other electrons. The wave function vanishes extremely rapidly if any two electrons approach each other, which helps to minimize the expectation value of the interaction.

How good the Laughlin wave function is? Let us see some numerics in Fig. 2, which compares the Laughlin wave function with numerically obtained exact diagonalization results.

Variational View Point

Alternatively, one can take q in Eq. 46 as a variational parameter. Then we can test which value of q will give the best variational energy. Consider Laughlins wave function as a function of the position z_k of some arbitrary but fixed electron k. There are $N-1$ factors of the type $(z_k - z_l)^q$, one for each of the remaining $N-1$ electrons. Now, remember that the highest power of the complex particle position is fixed by the number of states N_ϕ in each LL. This yields the relation $N_\phi = q(N-1)$. One notices that, in the thermodynamic limit, the "variational parameter" is entirely fixed by the filling factor $\nu = N/N_\phi = N/q(N-1) \approx 1/q$, thus $q = 1/\nu$. Since we need additional exchange relation, the even q should be excluded. Therefore, we only see FQHE at odd integer q.

Classical Plasma

We go further examine the physics behind this wave function. We think of the probability density arising from this wavefunction as if it were the Boltzmann weight for a problem in classical statistical mechanics. We define a fictitious inverse temperature $\beta = 1$ and classical Hamiltonian H_m via

$$|\psi_q|^2 = \exp[-H_m] \tag{47}$$

$$H_m = -2q \sum_{i<j} \ln |z_i - z_j| + \sum_j |z_j|^2/2\ell^2. \tag{48}$$

For a charge neutral two-dimensional classical plasma, the interaction is given by

$$V(r) = -e^2 \sum_{j<k} \ln r_{ij} + \frac{1}{2}\pi\rho e^2 \sum_j r_j^2 \tag{49}$$

where the particles are interacting via a two-dimensional Coulomb (logarithmic) interaction with each other and with a uniform neutralizing background. it is clear that H_m is the Hamiltonian for a two-dimensional classical plasma with, $e^2 = 2q, \rho_q = 1/(2\pi\ell^2 q)$. Therefore, in order to achieve charge neutrality, the plasma particles spread out uniformly in a disk with particle density corresponding to a filling factor $\nu = 1/q$, where q is an odd integer. The classical plasma provides strong support that the Laughlin state is indeed a translationally invariant liquid.

To interpret this form we should recall electrostatics in two dimensions: a point charge Q at the origin gives rise at radius r to an electric field

$$E(r) \sim \frac{Q}{2\pi r} withapotential V(r) \sim -\frac{Q}{2\pi} \ln r \tag{50}$$

Thus the two particle potential is like $-\frac{q^2}{2\pi} \ln |z_i - z_j|$ which represents the electrostatic interaction of particles with charge q.

The single particle term $\frac{q}{8\pi} \sum_k |z_k|^2$ would arise for particles of charge q moving in an electrostatic potential $|z|^2/(8\pi)$. We can view this potential as arising from a background charge distribution, and find the density of this charges using Poissons equation.

Quasi-hole Statistics and Fractionalization

We consider a situation where the filling factor is close to 1/q and there is only one quasihole in the system. Let us add a perturbation to a Hamiltonian which gives the fractional quantum Hall effect. The Hamiltonian then has a similar form to

$$H_{hole} = H_0 + \epsilon V(z - z_a) \tag{51}$$

The weak perturbation attracts the quasihole to z_a, which is a coordinate in the two-dimensional space represented by a complex number. The ground state of this Hamiltonian is evidently a state in which the quasihole is trapped at z_a:

$$\Psi(z_1, z_2, ..., z_{N_e}) \sim \prod_i (z_i - z_a)\psi_q \tag{52}$$

where ψ_q is the Laughlin wave function and there is an unspecified normalization factor here.

Now we move the quasihole in the real space and enclose a circle. The Berry phase in this case is calculated as follows:

$$\begin{aligned}
\gamma_C &= i\oint_C \langle \Psi_{r(t)}|\frac{\partial}{\partial r_t}\Psi_{r(t)}\rangle dr(t) \\
&= i\oint_C \langle \Psi_{z_a}|\frac{\partial}{\partial z_a}\Psi_{z_a}\rangle dz_a \\
&= i\oint_C dz_a \langle \Psi_{z_a}|[\frac{\partial}{\partial z_a}\sum_i \ln(z_i - z_a)]|\Psi_{z_a}\rangle \\
&= i\oint_C dz_a d^2 z \frac{\partial}{\partial z_a}\ln(z - z_a)\langle \Psi_{z_a}|\sum_i \delta(z - z_i)|\Psi_{z_a}\rangle \\
&= i\oint_C dz_a \int d^2 z \frac{\partial}{\partial z_a}\ln(z - z_a)\rho(z) \\
&= i\int d^2 z\rho(z)\oint_C dz_a \frac{\partial}{\partial z_a}\ln(z - z_a) \\
&= i\int d^2 z\rho(z)(-2i\pi) = 2\pi\int d^2 z\rho(z) = 2\pi\langle n\rangle = 2\pi S\frac{\nu}{2\pi\ell^2} = \frac{e}{q}\frac{BS}{\hbar}
\end{aligned} \tag{53}$$

Here we assume the electron density is uniform $\rho(z) = \nu/2\pi\ell^2$. This result can be interpreted as the AB phase that a quasihole of charge e/q acquires in the magnetic field. The size of this charge coincides with that of the quasi-hole at $\nu = 1/q$.

Similarly, we can also study statistics of the quasiparticles. When the two quasiholes are at z_a, z_b, the wave function can be written as

$$\Psi_{a,b} = \prod (z_i - z_a)(z_i - z_b)\psi_q(z_1, z_2, ..., z_{N_e}) \tag{54}$$

And we calculate the berry phase, when the quasihole at z_a moves adiabatically around a closed loop C. This calculation is parallel to the one quasihole case:

$$\gamma_C = 2\pi\int_C d^2 z\rho(z) = 2\pi\langle n\rangle \stackrel{!}{=} 2\pi(\frac{\nu}{2\pi\ell^2}S - \frac{1}{q}) \tag{55}$$

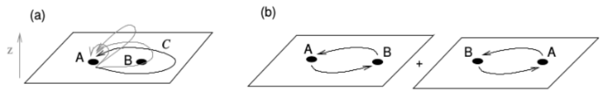

FIG. 3: (a) Process in which a particle A moves on a path C around another particle B. In three space dimensions, one may profit from the third direction (z-direction) to lift the path over particle B and thus to shrink the path into a single point. (b) Process equivalent to moving A on a closed path around B which consists, apart from a topologically irrelevant translation, of two successive exchanges of A and B.

Here the only difference is the density in the contour C is not uniform, but we have another quasihole at z_b, which leads to the density around z_b lower. The reduce of number of electrons is $1/q$. The charge of a quasihole should be independent of the presence of other quasiholes. So we must attribute this extra phase to some other cause. We interpret the extra phase as coming from a fictitious magnetic flux attached to each quasihole. Namely, we consider that the quasiholes are anyons by nature. Therefore, when we treat them as bosons in wavefunction, they appear as composite particles with flux attached to recover their anyonic nature: the exchange or interchange of two quasiholes gives a phase π/q, i.e. the quasi-holes obey fractional statistics.

Fractional Statistics

One of the most exotic consequences of charge fractionalisation in 2D quantum mechanics, exemplified by Laughlin quasi-particles, is fractional statistics. Remember that, in three space dimensions, the quantum-mechanical treatment of two and more particles yields a superselection rule according to which quantum particles are, from a statistical point of view, either bosons or fermions. This superselection rule is no longer valid in 2D (two space dimensions), and one may find intermediate statistics between bosons and fermions.

In order to illustrate the different statistical (i.e. exchange) properties of two quantum particles in three and two space dimensions, consider a particle A that moves adiabatically on a closed path C in the xy-plane around another one B of the same species (see Fig. 3). Path C in the xy-plane around another one B of the same species. We choose the path to be sufficiently far away from particle B and the two particles to be sufficiently localised such that we can neglect corrections due to the overlap between the two corresponding wave functions. Notice first that such a process T is equivalent to two successive exchange processes $T = E^2$.

Let us discuss first the three-dimensional case. Because of the presence of the third direction (z-direction), one may elevate the closed path in this direction while keeping the position of particle A fixed in the xy plane. We have $T(C) = 1$, so $E = 1$ (boson) or $E = -1$ (fermion).

In two space dimensions, this topological argument yields a completely different result. It is not possible to shrink a path C enclosing the second particle B into a single point at the position of A, without passing by B itself. From an algebraic point of view, the exchange processes are no longer described by the two roots of unity, 1 and -1, but by the so-called braiding group. In the simplest case of Abelian statistics,

$$\psi(\mathbf{r}_1)\psi(\mathbf{r}_2) = e^{i\alpha\pi}\psi(\mathbf{r}_2)\psi(\mathbf{r}_1) \tag{56}$$

where α is also called the statistical angle. One has $\alpha = 0$ for bosons and $\alpha = 1$ for fermions, and all other values of α in the interval between 0 and 2 for anyons. Sometimes anyonic statistics is also called fractional statistics - indeed all physical quasi-particles, such as those relevant for the FQHE, have an angle that is a fractional (or rational) number, but there is no fundamental objection that irrational values of the statistical angle should be excluded.

HOMEWORK: BOSON VERSUS FERMION

We consider a Hamiltonian for N-fermion:

$$H_F = \sum_{i=1}^{N} \frac{1}{2m}[\mathbf{p}_i^2 - e\mathbf{A}(\mathbf{r}_i)]^2 + \sum_i eA_0(\mathbf{r}_i) + \sum_{i<j} V(\mathbf{r}_i - \mathbf{r}_j) \tag{57}$$

$\mathbf{A}(\mathbf{r}_i)$ and A_0 are the vector potentials. The wave function satisfies the fermionic statistics,

$$H_F\Psi(\mathbf{r}_1, ...\mathbf{r}_N) = E\Psi(\mathbf{r}_1, ..., \mathbf{r}_N) \tag{58}$$

$$\Psi(\mathbf{r}_1, ...\mathbf{r}_i, ..., \mathbf{r}_j, ...\mathbf{r}_N) = -\Psi(\mathbf{r}_1, ...\mathbf{r}_j, ..., \mathbf{r}_i, ...\mathbf{r}_N) \tag{59}$$

Try to solve a flux attachment problem for bosons

$$H_B = \sum_{i=1}^{N} \frac{1}{2m}[\mathbf{p}_i^2 - e\mathbf{A}(\mathbf{r}_i) - e\mathcal{A}(\mathbf{r}_i)]^2 + \sum_i eA_0(\mathbf{r}_i) + \sum_{i<j} V(\mathbf{r}_i - \mathbf{r}_j) \tag{60}$$

which has a bosonic eigenvalue function

$$H_B\phi(\mathbf{r}_1, ...\mathbf{r}_N) = E\phi(\mathbf{r}_1, ...\mathbf{r}_N) \tag{61}$$

$$\phi(\mathbf{r}_1, ...\mathbf{r}_i, ..., \mathbf{r}_j, ...\mathbf{r}_N) = \phi(\mathbf{r}_1, ...\mathbf{r}_j, ..., \mathbf{r}_i, ...\mathbf{r}_N) \tag{62}$$

Please determine the form of required vector potential $\mathcal{A}$, and find the gauge transformation to prove the connection between H_F and H_B: $H_F = UH_BU^{-1}$, and $\Psi(\mathbf{r}_1, ...\mathbf{r}_N) = U\phi(\mathbf{r}_1, ...\mathbf{r}_N)$.

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[1] E. Fradkin, Field Theories of Condensed Matter Physics. Cambridge University Press, 2013.