Lecture note: Real space Renormalization Group

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The preceding chapters have shown that mean field theory does not accurately predict critical exponents, combined with some exact solutions. These parts are followed the same strategy: Statistical ensemble $->$ partition function or free energy $->$ thermal quantities (critical exponents). Next we will explore a different way to study the phase transitions.

In this chapter, we will explain how the scaling hypothesis follows from the presence of a diverging correlation length. The basic argument originates from the insight of L.P. Kadanoff that a diverging correlation length implies that there is a relationship between the coupling constants of an effective Hamiltonian and the length scale over which the order parameter is defined. Kadanoff's ingenious argument is correct in spirit, but not quite right in detail; as we will see, the relationship between coupling constants defined at different length scales is more complicated than assumed. Furthermore, Kadanoff's argument does not enable the critical exponents to be calculated. K.G. Wilson elaborated and completed Kadanoff's argument, showing how the relationship between coupling constants at different length scales could be explicitly computed, at least approximately; Wilson's theory, i.e. the renormalization group (RG), is thus capable of estimating the critical exponents. The RG also provides a natural framework in which to understand *universality.*

KADANOFF TRANSFORMATION AND COARSE-GRAINING PROCEDURE

As we have mentioned, the thermal quantities behave singularities close to the critical point. And the correlation length (which marks how far two spins are correlated) goes to infinity at the critical point ($\xi \to \infty$). [We didnot elaborate it explicitly. One can think about the 1d Ising chain again as an example: Spin correlation is $\langle \sigma_i \sigma_{i+r} \rangle = [\tanh \beta J]^r \equiv e^{-r/\xi}$, with $\xi = [\ln \cosh(\beta J)]^{-1} \approx e^{2\beta J} \to \infty$ for $T_c = 0$.] Another way to think about it is the computer simulation of 2D Ising model shown in Figure 2: At high T, the correlation length is small. Close (but above) T_c, somewhat larger clusters begin to develop. When the system reaches T_c, the clusters expand to infinite size, but fluctuations on smaller scale persist.

The property $\xi \to \infty$ is important, which motivates the Kadanoff's work on the critical point. Kadanoff has developed a different set of arguments which lead to the scaling ansatz for free energy density and correlation function and hence the scaling laws. The key assumption is that, under a length-scale coarse graining process, the partition function of the system satisfy

$$Z(J,N) = \sum_{\sigma_i} \exp(-\beta H[\sigma_i, K, N]$$
$$= \sum_{S_I} \exp(-\beta H[S_I, J_\ell, N\ell^{-d}] \equiv Z(J_\ell, N\ell^{-d}). \quad (1)$$

The example is the Ising model again and we imagine that the lattice is divided into cells as shown in Figure 1:

$$H = J \sum_{<ij>} \sigma_i \sigma_j - h \sum_i \sigma_i \quad (2)$$

The side length of a cell is ℓa where a is the lattice spacing. The parameter ℓ is chosen such that $a \ll \ell a \ll \xi$, the correlation length.

Then, we could imagine a coarse-graining procedure, in which we replace the spins within a block of side ℓa by a single spin, a block spin, which actually contains ℓ^d spins. The total number of blocks, and hence of block spins, is then N/ℓ^d. Now we will examine the consequences of such a coarse-graining procedure, which we will refer to for the moment as a block spin transformation.

It can be expected that, for $\xi \gg \ell a$, most of the spins in a particular cell are in the same direction. Further, the average spin parameters should be such that the interaction among them and with an external field yield the long range correlations existing in the original system. Thus, the attempt is to average out the short distance variations of the spins and make an Ising model with average spin parameters so that the new model has the same characteristics over long length scales. It is a hypothesis that a new model, satisfying these requirement, can be constructed.

We define the block spin S_I in block I by

$$S_I = \frac{1}{|m_\ell| \ell^d} \sum_{i \in I} \sigma_i, m_\ell = \frac{1}{\ell^d} \sum_{i \in I} \sigma_i \quad (3)$$

With this normalisation, the block spins S_i have the same magnitude as the original spins: $\langle S_I \rangle = \pm 1$.

Our assumption implies that we should define new coupling constants between the block spins and an effective external field which interacts with the block spins. We will denotes these respectively as J_ℓ and h_ℓ, with the subscript ℓ reminding us that in principle, these coupling constants depend upon the definition of the block spins, and hence depend upon

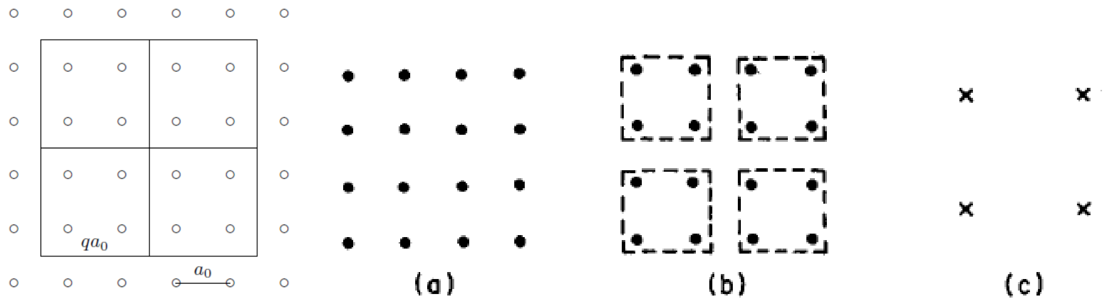

FIG. 1: Kadanoff's block-spin construction: (a) original lattice; (b) four spins grouped into a block; (c) the new lattice.

the coupling constants of the original Hamiltonian correspond to $\ell = 1$: $J_1 = J, h_1 = h$. In this regarding, the effective Hamiltonian for the block spins is given by

$$H_\ell = J_\ell \sum_{<IJ>} S_I S_J - h_\ell \sum_I S_I \tag{4}$$

which, by construction, is of the same form as the original Hamiltonian. The change is that the lattice spacing between the block spins is ℓa, whereas the spacing between the original spins is a; the former system also has fewer spins. Thus, for the block spins, the correlation length measured in units of the spacing ℓa of the block spins ξ_ℓ, is smaller than the correlation length ξ of the initial system, measured in units of the spacing a between the original spins:

$$\xi = \xi_\ell \times \ell a = \xi_1 a \to \xi_\ell = \xi_1/\ell \tag{5}$$

The system with Hamiltonian must be far away from criticality than the original system. Thus, we conclude that it is at a new effective reduced temperature, T_ℓ.

Similarly, the magnetic field h has been rescaled to an effective field h_ℓ, when measured in the appropriate units:

$$h \sum_i \sigma_i = h m_\ell \ell^d \sum_I S_I = h_\ell \sum_I S_I, \Rightarrow h_\ell = h m_\ell \ell^d \tag{6}$$

The effective Hamiltonian is of the same form as the original Hamiltonian, and thus the functional form of the free energy of the block spin system will be of the same form as that of the original system, albeit with J_ℓ and h_ℓ instead of J and h.

Scaling hypothesis

In terms of the free energies per spin or block spin,

$$N\ell^{-d}f_s(t_\ell, h_\ell) = Nf_s(t,h) \Rightarrow \ell^{-d}f_s(t_\ell, h_\ell) = f_s(t,h) \tag{7}$$

this equation describes how the free energy per spin transforms under a block spin transformation. We set $(t = T - T_c)$ here.

Since we seek to understand the power-law and scaling behaviour in the critical region, we assume that

$$t_\ell = t\ell^{y_T} \tag{8}$$

$$h_\ell = h\ell^{y_h} \tag{9}$$

The exponents y_T and y_h are assumed to be positive. Then

$$\ell^{-d}f_s(t\ell^{y_T}, h\ell^{y_h}) = f_s(t,h) \tag{10}$$

Since this condition should be satisfy at any ℓ, we thus have

$$\ell = |t|^{-1/y_T} \Rightarrow |t|^{d/y_T} f_s(1, h|t|^{-y_h/y_T}) = f_s(t,h) \equiv |t|^{2-\alpha}F_f(h/|t|^{\Delta}) \tag{11}$$

where $2-\alpha = \frac{d}{y_T}$ and $\Delta = y_h/y_T$, $F_f(x) = f_s(1,x)$.

Next we explore the relation of correlation function $G(x_\ell, t_\ell) =< S_I S_J > - < S_I >< S_J >$:

$$G(x_\ell, t_\ell) =< S_I S_J > - < S_I >< S_J > \tag{12}$$

$$= \frac{1}{\ell^{2(y_h-d)}\ell^{2d}} \sum_{i\in I, j\in J} < S_i S_j > - < S_i >< S_j > \tag{13}$$

$$= \frac{1}{\ell^{2(y_h-d)}\ell^{2d}} \ell^d \ell^d [< S_i S_j > - < S_i >< S_j >] \tag{14}$$

$$= \ell^{2(d-y_h)} G(x,t) \tag{15}$$

where we used $m_\ell = h_\ell \ell^{-d}/h = \ell^{y_h - d}$.

Using $x_\ell = x/\ell$ and setting $x_\ell = a$, we have

$$G(x/\ell, t\ell^{y}_{T}) = \ell^{2(d-y_h)}G(x,t) \Rightarrow G(x,t) = (\frac{x}{a})^{2(y_h-d)} G(a, t(\frac{x}{a})^{y_T}) \tag{16}$$

which gives the relation at the critical point $t = 0$:

$$G(x, t = 0) \sim (\frac{x}{a})^{2(y_h - d)} = \frac{1}{r^{d-2+\eta}} \tag{17}$$

where the critical exponent η is the critical exponent discussed before.

Kadanoff's block spin argument successfully motivates the functional form of the scaling relations; but it gives neither the exponents such as y_h, nor the scaling functions themselves. It does not address the issue of universality either, and as we have presented it, applies only to the Ising model, although it is clear that a generalisation to other systems is possible. The most crucial step is the assumption that the block spin Hamiltonian is of the same form as the original Hamiltonian.

Fixed points

The crucial ingredient of the RG method is the recognition of the importance and physical significance of fixed points of the RG transformation.

Let us suppose that we know the RG transformation $R_\ell[K]$. Then the fixed point of the RG transformation is a point $[K^*]$ in coupling constant space satisfying

$$[K^*] = R_\ell[K^*] \tag{18}$$

Now, under the RG transformation, length scales are reduced by a factor ℓ, as we have discussed. For any particular values of the coupling constants, we can compute the correlation length ξ, which transforms under R_ℓ according to the rule $\xi[K'] = \xi[K]/\ell$, indicating that the system moves further from criticality after RG transformation. At a fixed point,

$$\xi[K^*] = \xi[K^*]/\ell \tag{19}$$

which implies $\xi[K^*]$ can only be zero or infinity. We will refer to a fixed point with $\xi = \infty$ as a critical fixed point.

What can we learn from the behaviour of the flows near a fixed point? Let $K_n = K_n^* + \delta K_n$, so that the starting Hamiltonian is close to the fixed point Hamiltonian, and perform a RG transformation, then

$$K'_n = K'_n[K] = K_n^* + \delta K'_n = K_n^* + \sum_m \frac{\partial K'_n}{\partial K_m}|_{K_m = K_m^*} + ... \tag{20}$$

$M_{nm} = \frac{\partial K'_n}{\partial K_m}|_{K_m=K^*_m}$ is the linearized RG transformation in the vicinity of the fixed point K^*. The eigenvalues of M_{nm} determines the properties of the fixed point:

$$M_{nm}e^i_m = \Lambda^i e^i_n \tag{21}$$

where the eigenvalues Λ^i.

- $|\Lambda^i| > 1$: relevant means eigenvector/eigenvalue grows as RG process;
- $|\Lambda^i| < 1$: irrelevant means eigenvector/eigenvalue shrinks as RG;
- $|\Lambda^i| = 1$: marginal means eigenvector/eigenvalue doesnot change.

The significance of these distinctions is that if we start at K near K^*, but not on the critical manifold then the flows away from K^* i.e. in directions out of the critical manifold in the vicinity of K^*, are associated with relevant eigenvalues. The irrelevant eigenvalues correspond to directions of flow into the fixed point. The eigenvectors corresponding to the irrelevant eigenvalues span the critical manifold. The marginal eigenvalues turn out to be associated with logarithmic corrections to scaling, and are important at the upper and lower critical dimensions. The number of relevant eigenvalues must thus be the codimension c of the critical manifold, i.e., the difference between the dimensionalities of the coupling constant space and the critical manifold.

To sum up, we summary eight steps in real-space renormalization group:

- define block spins and related effective hamiltonain;
- trace over the original spins;
- obtain renormalization transformation;
- determine the fixed point and calculate the critical exponent.

Physically, ξ diverges at critical point leads to the conclusion that the system has no characteristic length, and is therefore invariant under scale transformations. Kadanoff's transformation is to enlarge the system size. The scale invariance means that there would no difference between the magnified part and the original system under the Kadanoff's transformation.

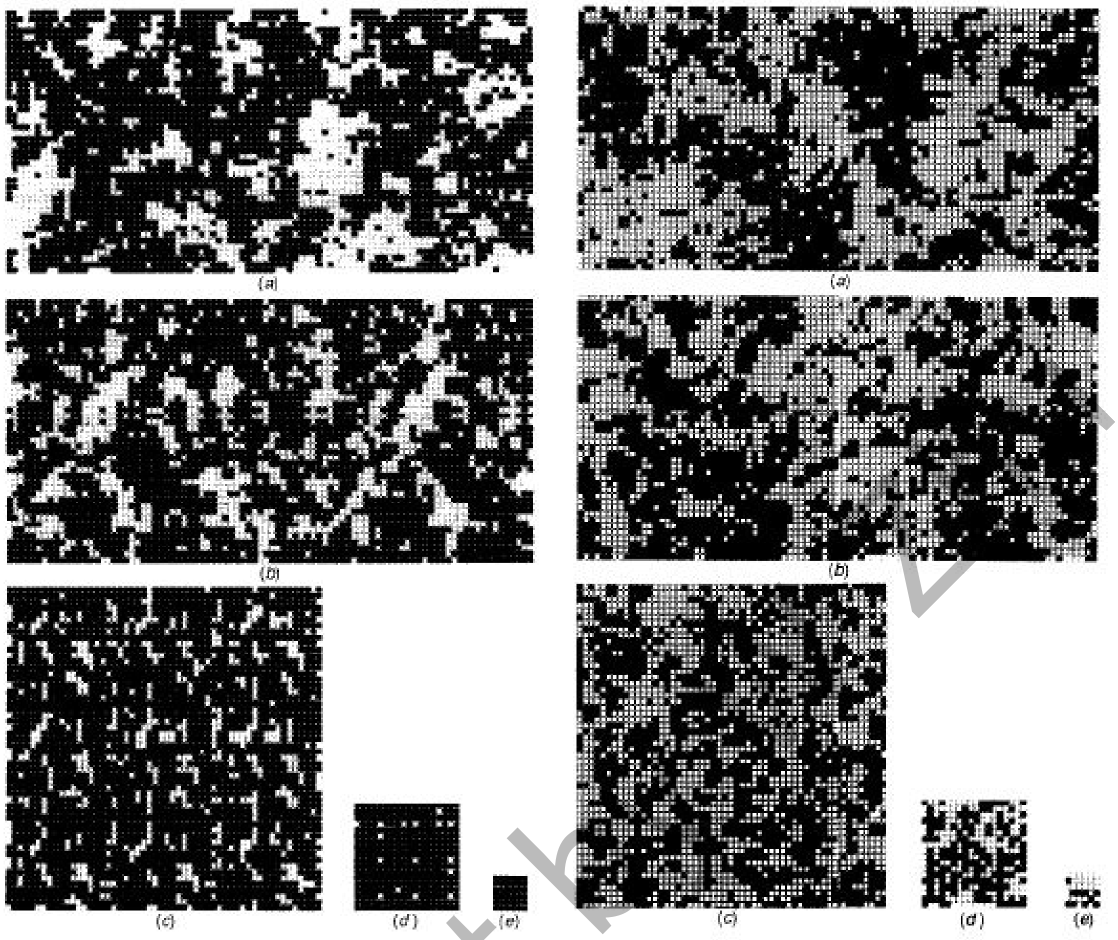

FIG. 2: (Left) Block-spin transformation is applied to a lattice repeatedly at $T = 0.99T_c$. Each time the number of spins is reduced by a factor of 9, elucidating the behaviour of the system at a larger scale. The correlation length decreases under successive transformations. Clearly seen is the decrease of the correlation length and a suppression of the fluctuations. Under each transformation the system appears more ordered, the system flows under the renormalization transformations towards zero temperature. (Right) Block-spin transformation at the critical point $T = T_c$, where the system remains critical. (K.G. Wilson, Sci. Am. 241, 140 (1979)

EXAMPLE: ONE-DIMENSIONAL ISING MODEL

An exact RG treatment can be carried out for the Ising model with nearest- neighbor interactions (Eq. 6.1) in one dimension. The basic idea is to find a transformation that reduces the number of degrees of freedom by a factor ℓ, while preserving the partition function, i.e.

$$Z = \sum_{\sigma_i} e^{-\beta H[\sigma_i]} = \sum_{\sigma'_i} e^{-\beta H[\sigma'_i]} \tag{22}$$

There are many mappings $\sigma_i \to \sigma'_i$ that satisfy this condition. The choice of the transformation is therefore guided by the simplicity of the resulting RG. With $\ell = 2$, for example, one possible choice is to group pairs of neighboring spins and define the renormalized spin as their average. This majority rule, $\sigma'_i = (\sigma_{2i-1} + \sigma_{2i})/2$, is in fact not very convenient as the new spin has three possible values $(0, 1, -1)$ while the original spins are two valued. There are several ways to deal with this problem. For example, the decimation method (as shown below), or we just choose three spins as a new unit cell.

Imagine grouping the spins by blocks of size 3, i.e. $(...[\sigma_1\sigma_2\sigma_3][\sigma_4\sigma_5\sigma_6]...)$. Each blocks may be in $2^3 = 8$ configurations. We can group these eight configurations in two disjoint sets to which we assign an effective spin σ'. We can for instance choose the majority rule so that $\sigma' = +$ if the three internal spins of the block are $[++-]$ or a permutation thereof, and $\sigma' = -$ if the internal spins are $[--+]$ up to permutation. Alternatively, it will actually be simpler to choose to assign to each block the spins of the middle site, so that the effective spin for the block $[\sigma_1\sigma_2\sigma_3]$ as $\sigma' = \sigma_2$. We then imagine computing the partition function in two steps: first summing over the internal spins of each blocks conditioned on their effective spins and second on the block effective spins.

Consider two adjacent blocks, e.g. $(...[\sigma_1\sigma_2\sigma_3][\sigma_4\sigma_5\sigma_6]...)$, the partition function will be

$$...e^{\beta J\sigma_1 s}e^{\beta J s\sigma_3}e^{\beta J\sigma_3\sigma_4}e^{\beta J\sigma_4 s'}e^{\beta J s'\sigma_6} \tag{23}$$

We sum over σ_3, σ_4 at s, s' fixed (the other spins σ_1, σ_6 is summed with other bonds). Using $e^{J\sigma\sigma'} = \cosh\beta J \times (1 + x\sigma\sigma')$ with $x = \tanh\beta J$, we may write this as the product of three

$$\begin{aligned}&\sum_{\sigma_3,\sigma_4} (\cosh\beta J)^3(1 + xs\sigma_3)(1 + x\sigma_3\sigma_4)(1 + x\sigma_4 s')\\ &= 4(\cosh\beta J)^3(1 + x^3 ss')\end{aligned} \tag{24}$$

Up to a multiplicative constant (independent of the spins) this expression, coding for the interaction between the effective blocks spins, is of the same form as that for the original spins but with a new interaction constant J'

$$x' = x^3 \Rightarrow \tanh\beta J' = (\tanh\beta J)^3 \tag{25}$$

The (new) hamiltonian for the block spin is thus identical to the original 1D Ising hamiltonian up to an irrelevant constant,

$$H'[s'] = Ne(J) - J' \sum_i s_i s_{i+1} \tag{26}$$

where $\ln e(J) = 2^{2/3}(\cosh \beta J)/(\cosh \beta J')^{1/3}$.

By iteration the effective coupling transforms as $x_{n+1} = x_n^3$ at each step. There is only two

fixed points: which gives the fixed points:

- $x^* = 0$ ($\beta = 0$): it is the sink for the disorder phase. If $x < 1$, $x_{n+1} \approx x_n^3 < x_n$ is even smaller, indicating that $x^* = 0$ is a stable fixed point with zero correlation length;
- $x^* = 1$ ($\beta = \infty$): it is the ordered phase. For a large but finite x, the renormalized interaction $x_{n+1} \approx x_n^3 < x_n$ is somewhat smaller. This fixed point is thus unstable.

Hence, the long distance degrees of freedom are effectively described by an infinite temperature: they are in the disordered paramagnetic phase (a statement that we already knew: no phase transition in 1D). The phase diagram is shown in Fig. 3.

The decimation method.— In the unit cell with two spins, we can assign one of the two spins, e.g. the even numbered spins, $s_i = \sigma_{2i}$. This RG procedure effectively removes the even numbered spins and is usually called a decimation:

$$Z = \sum_{\sigma_i} \exp[\sum_{i=1}^{N} B(\sigma_i, \sigma_{i+1})] = \sum_{\sigma_i, i=1}^{N/2} \sum_{s_i, i=1}^{N/2} \exp[\sum_{i=1}^{N/2} B(\sigma'_i, s_i) + B(s_i, \sigma'_{i+1})] \tag{27}$$

Summing over the decimated spins $\{s_i\}$, leads to

$$e^{-\beta H'(\sigma'_i)} = \prod_{i=1}^{N/2} [\sum_{s_i=\pm} e^{B(\sigma'_i, s_i) + B(s_i, \sigma'_{i+1})}] \equiv e^{\sum_{i=1}^{N/2} B'(\sigma'_i, \sigma'_{i+1})} \tag{28}$$

where we define

$$B(\sigma_1, \sigma_2) = \frac{h}{2}(\sigma_1 + \sigma_2) + K\sigma_1\sigma_2 \tag{29}$$

$$B'(\sigma'_1, \sigma'_2) = \frac{h'}{2}(\sigma'_1 + \sigma'_2) + K'\sigma'_1\sigma'_2 \tag{30}$$

Use the condition

$$\exp[K'\sigma_1'\sigma_2' + \frac{h'}{2}(\sigma_1' + \sigma_2')] = \sum_{s=\pm} \exp[Ks(\sigma_1' + \sigma_2') + \frac{h}{2}(\sigma_1' + \sigma_2') + hs] \tag{31}$$

, four possible configurations of the bond are

$$(+,+) : x'y' = y(x^2y + x^{-2}y^{-1}) \tag{32}$$

$$(-,-) : x'y'^{-1} = y^{-1}(x^{-2}y + x^2y^{-1}) \tag{33}$$

$$(+,-) : x'^{-1} = y + y^{-1} \tag{34}$$

$$(-,+) : x'^{-1} = y + y^{-1} \tag{35}$$

which gives

$$x' = \frac{1}{y + y^{-1}} \tag{36}$$

$$y' = (y + y^{-1})y(x^2y + x^{-2}y^{-1}) \tag{37}$$

where we have defined

$$x = e^K, y = e^h, x' = e^{K'}, y' = e^{h'} \tag{38}$$

For $h = 0$ ($y = 1$) case, we obtain that

$$e^{K'} = \frac{e^{2K} + e^{-2K}}{2} \Rightarrow 2K' = \ln\cosh(2K) \tag{39}$$

which gives the fixed points:

- $K^* = 0$: it is the sink for the disorder phase. If K is small, $K' \approx K^2$ is even smaller, indicating that $K^* = 0$ is a stable fixed point with zero correlation length;
- $K^* \to \infty$: it is the ordered phase. For a large but finite K, the renormalized interaction $K' \approx \ln(e^{2K}/2) \approx K - \ln 2/2$ is somewhat smaller. This fixed point is thus unstable with an infinite correlation length.

Clearly any finite interaction renormalizes to zero, indicating that the one- dimensional chain is always disordered at sufficiently long length scales.

FIG. 3: Phase diagram of 1D Ising model.

EXAMPLE: TWO-DIMENSIONAL TRIANGULAR LATTICE

Let us treat the Ising model on a triangular lattice, subject to the usual nearest-neighbor Hamiltonian

$$H = -J \sum_{<ij>} \sigma_i \sigma_j \tag{40}$$

The original lattice sites are grouped into cells of three spins (e.g. in alternating up pointing triangles). Labeling the three spins in a block cell α as $\{\sigma_\alpha^1, \sigma_\alpha^2, \sigma_\alpha^3\}$, we can use a "majority rule" to define the renormalized cell spin as

$$\sigma'_\alpha \equiv sign[\sigma_\alpha^1, \sigma_\alpha^2, \sigma_\alpha^3] \tag{41}$$

The distance between new block spin becomes $L = \sqrt{3}a$. The new block spins make up a new triangular lattice, with the same symmetry and number of nearest neighbors. It is equivalent to enlarge the unit cell by L times, which is a coarse-graining process.

The block spin can take two values: ± 1. $\sigma_\alpha = 1(-1)$ relates to four different possibilities, as shown in Tab. I.

The renormalized interactions corresponding to the above map are obtained from the constrained sum

$$e^{-\beta H'[\sigma'_\alpha]} = \sum_{\{\sigma_i\}} e^{-\beta H[\sigma_i]} \tag{42}$$

Let us write the original Hamiltonian as two parts: intra-unit-cell part and inter-unit-cell part: $H = H_0 + V$. The intra-part is easy to get:

$$H_0 = -J \sum_\alpha \sigma_\alpha^1 \sigma_\alpha^2 + \sigma_\alpha^1 \sigma_\alpha^3 + \sigma_\alpha^2 \sigma_\alpha^3 \tag{43}$$

$$\rightarrow Z_0 = \prod_\alpha e^{\beta J(\sigma_\alpha^1 \sigma_\alpha^2 + \sigma_\alpha^1 \sigma_\alpha^3 + \sigma_\alpha^2 \sigma_\alpha^3)} = (e^{3K} + 3e^{-K})^{N/3} \tag{44}$$

which is just a number independent of block spins.

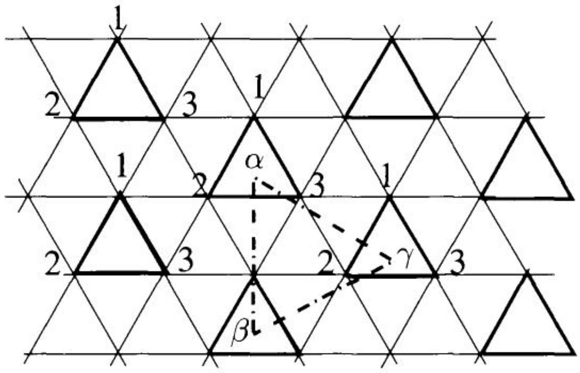

FIG. 4: Block spin of 2D Ising model on the triangular lattice.

To calculate the partition function, we can follow the definition as

$$Z = \sum_{\{\sigma_i\}} e^{-\beta H[\sigma_i]} = \sum_{\{\sigma_i\}} e^{-\beta H_0 - \beta V} = \sum_{\{\sigma_i\}} Z_0 \frac{e^{-\beta H_0 - \beta V}}{Z_0} \equiv Z_0 < e^{-\beta V} > \tag{45}$$

where $< e^{-\beta V} >$ stands for the thermal average of $e^{-\beta V}$ over $e^{-\beta H_0}$. Next we try to solve $< e^{-\beta V} >$, but only the approximated value can be obtained analytically. We use the Taylor expansion $\ln(1+x) \approx x - \frac{x^2}{2} + \frac{x^3}{3} ...$, we have (by assuming $< V >$ is a small number)

$$\ln Z = \ln(Z_0) + \ln < e^{-\beta V} > = \ln(Z_0) + \ln(1 + < -\beta V > + ...) \approx \ln(Z_0) + < -\beta V > + ... \tag{46}$$

The interaction term between block spin α and β is

$$< -\beta V > = \beta J \sum_{<\alpha\beta>} [< \sigma_\alpha^1 \sigma_\beta^2 > + < \sigma_\alpha^1 \sigma_\beta^2 >] = \beta J \sum_{<\alpha\beta>} [< \sigma_\alpha^1 >< \sigma_\beta^2 > + < \sigma_\alpha^1 >< \sigma_\beta^2 >] \tag{47}$$

For the case of block spin $\sigma_\alpha = +1$, we have

$$< \sigma_\alpha^1 > = \frac{\sum_{\sigma_\alpha^{i=1,2,3}} \sigma_\alpha^1 e^{-\beta H_0}}{\sum_{\sigma_\alpha^{i=1,2,3}} e^{-\beta H_0}} = \frac{e^{3\beta J} - e^{-\beta J} + 2e^{-\beta J}}{e^{3\beta J} + 3e^{-\beta J}} = \frac{e^{3\beta J} + e^{-\beta J}}{e^{3\beta J} + 3e^{-\beta J}} \tag{48}$$

For the case of block spin $\sigma_\alpha = -1$, we have

$$< \sigma_\alpha^1 > = \frac{\sum_{\sigma_\alpha^{i=1,2,3}} \sigma_\alpha^1 e^{-\beta H_0}}{\sum_{\sigma_\alpha^{i=1,2,3}} e^{-\beta H_0}} = \frac{-e^{3\beta J} + e^{-\beta J} - 2e^{-\beta J}}{e^{3\beta J} + 3e^{-\beta J}} = -\frac{e^{3\beta J} + e^{-\beta J}}{e^{3\beta J} + 3e^{-\beta J}} \tag{49}$$

Then, by combining the above two cases, we have

$$< \sigma_\alpha^1 > = \frac{e^{3\beta J} + e^{-\beta J}}{e^{3\beta J} + 3e^{-\beta J}} \sigma_\alpha \tag{50}$$

TABLE I: Values of spins in each block spin.

σ'_α	σ^1_α	σ^2_α	σ^3_α	$e^{-\beta H_0}$
+1	+1	+1	+1	$e^{3\beta J}$
+1	-1	+1	+1	$e^{-\beta J}$
+1	+1	-1	+1	$e^{-\beta J}$
+1	+1	+1	-1	$e^{-\beta J}$
-1	-1	-1	-1	$e^{3\beta J}$
-1	-1	-1	+1	$e^{-\beta J}$
-1	-1	+1	+1	$e^{-\beta J}$
-1	+1	-1	-1	$e^{-\beta J}$

And we can get the same relations for $<\sigma^2_\alpha>, <\sigma^3_\alpha>$. And plugging in these relations back to $<\beta V>$, we get

$$
\begin{aligned}
<-\beta V> &= \beta J \sum_{<\alpha\beta>} [<\sigma^1_\alpha><\sigma^2_\alpha> + <\sigma^1_\alpha><\sigma^2_\alpha>] = 2\beta J \sum_{<\alpha\beta>} [\frac{e^{3\beta J}+e^{-\beta J}}{e^{3\beta J}+3e^{-\beta J}}]^2 \sigma_\alpha\sigma_\beta \\
&\equiv \beta J' \sum_{<\alpha\beta>} \sigma_\alpha\sigma_\beta \qquad (51)
\end{aligned}
$$

which leads to

$$
\beta J' = 2\beta J [\frac{e^{3\beta J}+e^{-\beta J}}{e^{3\beta J}+3e^{-\beta J}}]^2 \qquad (52)
$$

⋄ *H*omework: Please analyze the fixed points of the above RG equation. How many fixed points are there? What is the difference from the 1D Ising model?