Table of Contents

Chapter 1

Renormalization Group

" The basic problem causing the difficulties in understanding critical phenomena is the problem of the infinite number of degrees of freedom. This problem is also the bottleneck in quantum field theory and in many of the stubborn problems in solid-state physics."— K. G. Wilson, 1971

In this chapter, we will explore the renormalization group based on the field action in the continuum space. In history, the renormalization group was initialized in the theory of fundamental interactions at the microscopic scale, and then the theory of continuous macroscopic phase transitions. In the former framework, it emerged as a consequence of the necessity of renormalization to cancel infinities that appear in a straight-forward interpretation of quantum field theory, and of the freedom of then defining the parameters of the renormalized theory at different momentum scales. In the statistical physics of phase transitions, a more general renormalization group, based on a recursive averaging over short distance degrees of freedom, was later introduced to explain the universal properties of contin- uous phase transitions.

The idea of renormalization group is quite intuitive, i.e. the course graining process offers a unique way to capture the physics around the critical point. One could introduce how to implement this idea in the real-space, by enlarge the unit cell by a factor $1 < \ell < \xi$, dubbed as the real-space renormalization group. Alternatively, this course graining process can be performed in the momentum space (see Fig. 1.1), which was first introduced by K. G. Wilson.

The general procedure in the momentum space is

- Eliminate fast modes, i.e. reduce the cut-off from Λ to Λ/s

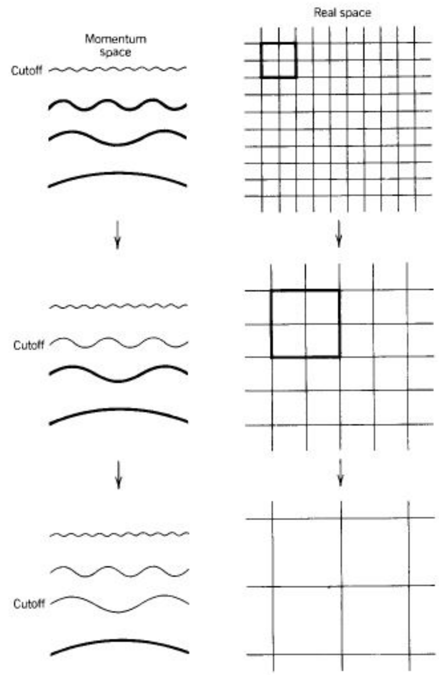

Figure 1.1: Coarse-graining in momentum space and in real space. In the former, one effectively lowers the cutoff. In the latter, one blots out finer details, enlarging the effective lattice spacing.

- Introduce rescaled momenta, $k' = sk$, which shifts the momentum back to Λ
- Introduce resclaed fields $\phi'(k') = \phi(k'/s)$ and express the effective action in terms of them

1.1 Statistical field theory

In the theory of continuous phase transitions, one is interested in the large distance behaviour or macroscopic properties of physical observables near the transition temperature $T = T_c$. At the critical temperature, the correlation length, the length scale on which correlations above T_c decay exponentially, diverges. This gives rise to non-trivial large distance properties that are, to a large extent, independent of the short distance structure, a property called universality.

Intuitive arguments indicate that, when the correlation length is large, the large distance behaviour can be inferred from a statistical field theory in continuum space (short-ranged correlation is irrelevant to the long-distance physics). Therefore, we consider a classical statistical system defined in terms of a "field" $\phi(x)$ in continuum space, $x \in R^d$, and a functional measure on fields of the form $\exp(-H[\phi])/Z$, where $H[\phi]$ is called the Hamiltonian in statistical physics and Z is the partition function (a normalization) given by the field integral

$$Z = \int D[\phi] e^{-H[\phi]} \tag{1.1}$$

where the dependence in the temperature T is included in $H[\phi]$. The essential condition of short range interactions in the initial statistical system translates into the property of locality of the field theory: $H[\phi]$ can be chosen as a space-integral over a linear combination of monomials in the field and its derivatives.

As an example, we write down the so-called ϕ^4 theory describing the critical phenomenon. There are two ways to think about it. The first, starting from the Landau mean-field free-energy, and we "enhance" it to a functional form. In this regarding, we just replace the order parameter, say m, by a function $\phi(x)$. The spatial dependence describes the fluctuation in the real space. The second method is to derive it from the Ising model (as shown in the frame box). The action reads:

$$Z = \int D[\phi] \exp[-\int d^d x(-\nabla\phi)^2 + r\phi^2 + u\phi^4)] \tag{1.2}$$

Effective field theory of the Ising model.— The long-range character of critical fluctuations and the universality of the critical properties suggest that it is possible to calculate the critical behavior by a phenomenological field theory rather than a microscopic model. One may neglect the lattice completely and describe the partition function of the system near T_c by a functional integral over a continuous local order field $\phi(x)$. The energy functional is assumed to have a Taylor expansion in the order field $\phi(x)$ and in its gradients. Here we derive this phenomenological theory.

Let us start from the partition function

$$
\begin{aligned}
Z = & \sum_{\sigma_1=\pm,\dots,\sigma_N=\pm} \exp[\beta \sum_{\langle ij\rangle} \sigma_i \sigma_j] = \sum_{\{\sigma_i\}} \exp[\sum_{\langle ij\rangle} \sigma_i K_{ij} \sigma_j] \\
& = const. \int d\mathbf{v} \sum_{\{\sigma_i\}} e^{-\frac{1}{2}\sum_{ij} v_i (K^{-1}_{ij}) v_j + \sum_i \sigma_i v_i} \\
& = const. \int D[v] e^{-\frac{1}{2}\sum_{ij} v_i (K^{-1}_{ij}) v_j} \prod_i (2\cosh(v_i)) \\
& \overset{\phi_i = \sum_j K^{-1}_{ij} v_j}{=} const. \int D\phi e^{-\sum_{ij}\phi_i K_{ij}\phi_j + \sum_i \ln\cosh(2\sum_j K_{ij}\phi_j)} = \int D\phi e^{-S[\phi]}
\end{aligned}
\tag{1.3}
$$

where we introduce an auxiliar field v_i and we used the Gaussian integral

$$
\int d\mathbf{v} e^{-\frac{1}{2}\vec{v}\cdot A \cdot \vec{v} + \vec{v}\cdot\vec{j}} = (2\pi)^{N/2} [detA]^{-1/2} e^{\frac{1}{2}\vec{v}\cdot\vec{j}}
\tag{1.4}
$$

Then we assume the field is small $|\phi(x)| \ll 1$, and the spatial profile of the field is smooth, we can expand $\ln\cosh(x) = x^2/2 - x^4/12 + ...$ If writing in the Fourier space, $\phi_i = \frac{1}{N}\sum_k e^{i\mathbf{k}\cdot\mathbf{r}_i}\phi(\mathbf{k})$ and $K_{ij} = \frac{1}{N}\sum_k e^{i\mathbf{k}\cdot(\mathbf{r}_i - \mathbf{r}_j)} K(\mathbf{k})$, we obtain that

$$
S[\phi] = \sum_k \phi_{-\mathbf{k}} [c_0 + c_2 k^2] \phi_{\mathbf{k}} + c_4 \sum_{k_1,k_2,k_3,k_4} \delta_{\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4=0} \phi_{\mathbf{k}_1}\phi_{\mathbf{k}_2}\phi_{\mathbf{k}_3}\phi_{\mathbf{k}_4} + ...
\tag{1.5}
$$

where we used $K(k) = K(0) + \frac{1}{2}k^2 K''(0) + ...$ If write in the spatial space, we have

$$
S[\phi] = \int d^d x (c_2 (\nabla\phi)^2 + c_0 \phi^2 + c_4 \phi^4)
\tag{1.6}
$$

This is the so-called ϕ^4 theory or action. It is a phenomenogical theory of Ising model.

Also, this functioanl is also called the Ginzburg-Landau functional. It differs from Landaus expansion of the Gibbs free energy by the first gradient term (apart from the more general notation).

Therefore, in the framework of statistical field theories relevant for simple phase transitions, we explain first the perturbative renormalization group. We then review a few important applications like the proof of scaling laws and the determination of singularities of thermodynamic functions at the transition.

1.2 Gaussian Fixed Point

Let us recall the effective ϕ^4 theory theory of Ising model:

$$S[\phi] = \int d^d x((\nabla\phi)^2 + r\phi^2 + u\phi^4) \tag{1.7}$$

When $u = 0$, this is called Gaussian model and it is solvable. Please note that, here $\phi(x)$ is a field describing the physics around the critical point. Instead of removing fluctuations at scales $a < x < sa$, we now remove Fourier modes in the momentum shell $\Lambda/s < q < \Lambda$ to implement the coarse-graining.

The partition function of the Gaussian model (in momentum space) is

$$Z = \int [D\phi(k)]e^{-S[\phi]} = \int [D\phi(k)] \exp[-\int_0^\Lambda \frac{d^d k}{(2\pi)^d}\phi(-\mathbf{k})k^2\phi(\mathbf{k})] \tag{1.8}$$

We will study $\Lambda \sim 1/a$ centered at the origin of BZ and ignore the shape of the BZ.

The dimension d is usually an integer, but we mush be prepared to work with continuous d. For our discussions, we need the integration measure for rotationally invariant integrands:

$$d^d k = k^{d-1} dk S_d, \quad S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)} \tag{1.9}$$

At the first step, we divide the field into slow and fast modes:

$$\phi(k) = \begin{cases} \phi_s(\mathbf{k}), 0 < |\mathbf{k}| < \Lambda/s \\ \phi_f(\mathbf{k}), \Lambda/s < |\mathbf{k}| < \Lambda \end{cases} \tag{1.10}$$

where $s > 1$ is the scale parameter.

The action itself becomes

$$S[\phi] = S_{\text{slow}} + S_{\text{fast}} = \int_0^{\Lambda/s} \frac{d^d k}{(2\pi)^d}\phi_s(-\mathbf{k})k^2\phi_s(\mathbf{k}) + \int_{\Lambda/s}^{\Lambda} \frac{d^d k}{(2\pi)^d}\phi_f(-\mathbf{k})k^2\phi_f(\mathbf{k}) \tag{1.11}$$

Thus integrating over the fast modes just gives an overall constant multiplying the Z for the slow modes:

$$
\begin{aligned}
Z &= \int [D\phi_s(k)] e^{-S[\phi_s]} \int [D\phi_f(k)] e^{-S[\phi_f]} \\
&= \int [D\phi_s(k)] e^{-S[\phi_s] + \ln Z_{fast}} \qquad (1.12) \\
Z_{fast} &= \int [D\phi_f(k)] e^{-S[\phi_f]} \qquad (1.13)
\end{aligned}
$$

We ignore Z_{fast} because it is independent of ϕ_s.

At the second step, the action after mode elimination followed by a new momentum $k' = sk$, then

$$
\begin{aligned}
S[\phi] &= \int_0^{\Lambda/s} \frac{d^d k}{(2\pi)^d} \phi_s(-k) k^2 \phi_s(k) \\
&= \int_0^{\Lambda} \frac{d^d k'}{s^d (2\pi)^d} \phi_s(-k'/s) k'^2/s^2 \phi_s(k'/s) \\
&= s^{-(d+2)} \int_0^{\Lambda} \frac{d^d k'}{(2\pi)^d} \phi_s(-k'/s)^2 k^2 \phi_s(k'/s) \qquad (1.14)
\end{aligned}
$$

At the third step, to take care of the factor $s^{-(d+2)}$ we introduce a rescaled field

$$\phi'(k') = s^{-(d/2+1)} \phi_s(k'/s) \qquad (1.15)$$

and we get

$$S[\phi'] = \int_0^{\Lambda} \frac{d^d k'}{(2\pi)^d} \phi'(-k') k'^2 \phi'(k') \qquad (1.16)$$

With this definition of the RG transformation, we have a mapping from actions defined in a certain k-space to actions in the same space. Thus, if we represent the initial action as a point in a coupling constant space, this point will flow under the RG transformation to another point in the same space. If we make a connection with the real-space RG, the first step is make a bulk spin, the second step is to do the summation and to construct the effective model, and the third step is to rescale (renormalize) the field.

We now want to determine the flow of couplings near this fixed point. We will do this by adding some perturbations in linear, quadratic and quartic in ϕ.

1.2.1 linear operator

The uniform magnetic field couples linearly to ϕ as $h\phi(k=0)$. When defining the rescaled field, we see

$$h\phi(0) = hs^{(d/2+1)}\phi'(0\cdot s) = h_s\phi'(0), \quad h_s = hs^{1+d/2} \tag{1.17}$$

Here we see an infinitesimal h gets amplified by the RG process, so h is relevant parameter.

Comparing it with the scaling hyphethesis, we see $y_h = 1 + d/2$ for Gaussian field theory.

1.2.2 quadratic operator

$$\begin{aligned}
S_r &= \int_0^{\Lambda} \frac{d^dk}{(2\pi)^d}\phi(-\mathbf{k})r\phi(\mathbf{k}) \\
&= const. \int_0^{\Lambda/s} \frac{d^dk}{(2\pi)^d}\phi_s(-\mathbf{k})r\phi_s(\mathbf{k}) = const. \int_0^{\Lambda} \frac{d^dk'}{s^d(2\pi)^d}\phi_s(-\mathbf{k}'/s)r\phi_s(\mathbf{k}'/s) \\
&= s^2\int_0^{\Lambda} \frac{d^dk}{(2\pi)^d}\phi'(-\mathbf{k}')r\phi(\mathbf{k}') \equiv \int_0^{\Lambda} \frac{d^dk}{(2\pi)^d}\phi'(-\mathbf{k}')r_s\phi(\mathbf{k}') \\
&\Rightarrow r_s = rs^2
\end{aligned} \tag{1.18}$$

Again, we have identified another relevant parameter in r.

Comparing the scaling of $r_s = rs^2$ with the scaling hyphothesis, we can infer $y_t = 2$ (please note that $r \sim a(T) \propto (T - T_c) = t$). So we may identify r with t, the dimensionless temperature that takes us off criticality.

Since the correlation length ξ drops by a factor of $1/s$ under this rescaling of momentum by s, we have

$$\begin{aligned}
&\xi(r)/s = \xi(r_s) \\
&\Rightarrow \xi(r) = s\xi(rs^2) = r^{-\frac{1}{2}}\xi(1) \Rightarrow \nu = \frac{1}{2}
\end{aligned} \tag{1.19}$$

Thus, we have $y_t = 2$ and $y_h = 1 + d/2$, $\nu = 1/2$, from which we can calculate the critical exponents (see discussion below)

$$\alpha = \frac{4-d}{2}, \beta = \frac{d-2}{4}, \gamma = 1, \delta = \frac{d+2}{d-2}, \nu = \frac{1}{2}, \eta = 0 \tag{1.20}$$

1.3 Wilson-Fisher critical point

Let us consider a perturbation around the Gaussian point. This will lead to a new fixed point. When we consider quartic perturbations of the fixed point, we run into a new complication: the term couples slow and fast modes and we have to do more than just rewrite the old perturbation in terms of new fields and new momenta.

1.3.1 Quartic operator (First order)

$$\begin{aligned} S = S_0 + S_u = & \int_0^\Lambda \frac{d^d k}{(2\pi)^d} \phi(-\mathbf{k})(k^2 + r)\phi(\mathbf{k}) \\ & + \frac{u}{4} \int_{|k|<\Lambda} \frac{d^d k_{i=1,2,3,4}}{(2\pi)^d} \phi(\mathbf{k}_4)\phi(\mathbf{k}_3)\phi(\mathbf{k}_2)\phi(\mathbf{k}_1)\delta(\sum_i \mathbf{k}_i) \end{aligned} \tag{1.21}$$

Integrating over fast modes:

$$\begin{aligned} Z &= \int [D\phi_s] e^{-S_0[\phi_s]} \int [D\phi_f] e^{-S_0(\phi_f)} e^{-S_u[\phi_s, \phi_f]} \\ &= \int [D\phi_s] e^{-S_{eff}[\phi_s]} \end{aligned} \tag{1.22}$$

The effective action is

$$\begin{aligned} e^{-S_{eff}[\phi_s]} &= e^{-S_0[\phi_s]} \int [D\phi_f] e^{-S_0(\phi_f)} e^{-S_u[\phi_s,\phi_f]} \\ &= e^{-S_0[\phi_s]} \frac{\int [D\phi_f] e^{-S_0(\phi_f)} e^{-S_u[\phi_s,\phi_f]}}{\int [D\phi_f] e^{-S_0[\phi_f]}} \int [D\phi_f] e^{-S_0[\phi_f]} \\ &= e^{-S_0[\phi_s]} \langle e^{-S_u[\phi_s,\phi_f]} \rangle_f \\ &= e^{-[S_0[\phi_s] + \langle S_u[\phi_s,\phi_f]\rangle - \frac{1}{2}[\langle S_u[\phi_s,\phi_f]^2\rangle - \langle S_u[\phi_s,\phi_f]\rangle^2 + \ldots]} \equiv e^{-S_{eff}[\phi]} \end{aligned} \tag{1.23}$$

Here we utilize the cumulant expansion, which relates the mean of the exponential to the exponential of the means,

$$\langle e^{\Omega}\rangle = e^{\langle\Omega\rangle+\frac{1}{2}[\langle\Omega^2\rangle-\langle\Omega\rangle^2]+...} \tag{1.24}$$

and

$$S_{eff} \approx S_0 + \langle S_u\rangle - \frac{1}{2}[\langle S_u^2\rangle - \langle S_u\rangle^2] + ... \tag{1.25}$$

The leading term in the cumulant expansion has the form

$$\langle S_u\rangle = \frac{u}{4}\langle\int_{k<\Lambda}(\phi_f+\phi_s)_4(\phi_f+\phi_s)_3(\phi_f+\phi_s)_2(\phi_f+\phi_s)_1\rangle \tag{1.26}$$

There are 16 possibilities falling into four groups: 8 terms with an odd number of fast modes, 1 term with all fast modes, 1 term with all slow modes, and 6 terms with two slow and two fast modes. The terms with odd number of fast modes donot contribute the action, we ignore them. The six terms of two fast modes are our focus, i.e. $\phi_f\phi_f\phi_s\phi_s$, $\phi_s\phi_s\phi_f\phi_f$, $\phi_f\phi_s\phi_s\phi_f$, $\phi_s\phi_f\phi_f\phi_s$, $\phi_s\phi_f\phi_s\phi_f$, $\phi_f\phi_s\phi_f\phi_s$.

So we need calculate

$$\begin{aligned}
\langle S_u\rangle &= 6\frac{u}{4}\int \phi_s(k_4)\phi_s(k_2)\langle\phi_f(k_3)\phi_f(k_1)\rangle(2\pi)^d\delta(k_4+k_3+k_2+k_1)\frac{d^dk}{(2\pi)^d}\\
&= \int_0^{\Lambda/s}\phi_s(-k)\phi_s(k)\frac{d^dk}{(2\pi)^d}[6\frac{u}{4}\int_{\Lambda/s}^{\Lambda}\frac{d^dk_3}{(2\pi)^d}\frac{1}{k_3^2+r}]\\
&= \int_0^{\Lambda/s}\phi_s(-k)\phi_s(k)\frac{d^dk}{(2\pi)^d}[6\frac{u}{4}\frac{S_d}{(2\pi)^d}\frac{\Lambda^d}{\Lambda^2+r}dl],
\end{aligned} \tag{1.27}$$

where we used $s = e^l \approx 1 + dl$ and $dl \ll 1$. In the last line, we have done the integral with special treatment

$$\begin{aligned}
&\int_{\Lambda/s}^{\Lambda}\frac{d^dk}{(2\pi)^d}\frac{1}{k^2+r}=\int_{\Lambda/s}^{\Lambda}\frac{S_dk^{d-1}dk}{(2\pi)^d}\frac{1}{k^2+r}\\
&\approx\frac{S_d\Lambda^{d-2}}{(2\pi)^d}\int_{\Lambda/s}^{\Lambda}\frac{kdk}{k^2+r}\\
&=\frac{S_d\Lambda^{d-2}}{(2\pi)^d2}[\ln(\Lambda^2+r)-\ln(\Lambda^2/s^2+r)]\\
&\approx\frac{S_d\Lambda^{d-2}}{(2\pi)^d2}[\ln(\Lambda^2+r)-\ln(\Lambda^2(1-2dl)+r)]\\
&=\frac{S_d\Lambda^{d-2}}{(2\pi)^d2}[\ln(\Lambda^2+r)-\ln[(\Lambda^2+r)(1-\frac{2dl\Lambda^2}{\Lambda^2+r})]]\\
&\approx\frac{S_d\Lambda^{d-2}}{(2\pi)^d2}\frac{2dl\Lambda^2}{\Lambda^2+r}=\frac{S_d}{(2\pi)^d}\frac{\Lambda^d}{\Lambda^2+r}dl
\end{aligned}\tag{1.28}$$

And the two-point correlation function is equivalent to the Green's function. Or, one can deduce it based on the Gaussian integeral as shown in the box below.

A simple way to get the two-point correlation function.—

Let us recall the simplest Gaussian integral for a variable ϕ:

$$\begin{aligned}
&\int d\phi e^{-\frac{K}{2}\phi^2+h\phi}=\sqrt{\frac{2\pi}{K}}e^{\frac{h^2}{2K}}\\
&\Rightarrow\frac{\partial^2}{\partial h}\int d\phi e^{-\frac{K}{2}\phi^2+h\phi}=\sqrt{\frac{2\pi}{K}}e^{\frac{h^2}{2K}}[\frac{1}{K}+\frac{h^2}{K^2}]
\end{aligned}\tag{1.29}$$

When we change the integral for a variable to functional integral for a function, the very similar result can be obtained:

$$\begin{aligned}
\langle\phi_i\phi_j\rangle&=\int D[\phi]\phi_i\phi_je^{-\sum_{ij}-\frac{K_{ij}}{2}\phi_i\phi_j}=[\int D[\phi]\frac{\delta}{\delta h_i}\frac{\delta}{\delta h_j}e^{-\sum_{ij}-\frac{K_{ij}}{2}\phi_i\phi_j+h_i\phi_i}]|_{h\to0}\\
&=[\frac{\delta}{\delta h_i}\frac{\delta}{\delta h_j}\sqrt{\frac{(2\pi)^N}{det(K)}}\exp[\sum_{i,j}\frac{K_{ij}}{2}h_ih_j]]|_{h\to0}\\
&=\sqrt{\frac{(2\pi)^N}{det(K)}}(K_{ij})^{-1}
\end{aligned}\tag{1.30}$$

If working in the momentum space, the correlation function is diagonal, so we have

$$\langle \phi(\mathbf{k})\phi(\mathbf{k}')\rangle = \int D[\phi]\phi(\mathbf{k})\phi(\mathbf{k}')e^{-\int_0^\Lambda \frac{d^dk}{(2\pi)^d}\phi(-\mathbf{k})(k^2+r)\phi(\mathbf{k})} = \frac{1}{k^2+r}\delta(\mathbf{k}+\mathbf{k}') \tag{1.31}$$

We then make the rescaling

$$k' = sk, \quad \phi'(k') = s^{-(d/2+1)}\phi(k'/s) \tag{1.32}$$

we have

$$\langle S_u\rangle = s^2\int_0^\Lambda \phi(-k')\phi(k')\frac{d^dk'}{(2\pi)^d}[uAdl] + s^{4-d}\frac{u}{4}\int_{k<\Lambda}\prod_{i=1}^{3}\frac{d^dk_i'}{(2\pi)^d}\phi(k_4')\phi(k_3')\phi(k_2')\phi(k_1')\delta(4+3+2+1) \tag{1.33}$$

We obtain the term like

$$(1+2dl)\int_0^\Lambda \phi(-k)\phi(k)(r+Audl)\frac{d^dk}{(2\pi)^d} = \int_0^\Lambda r'\phi(-k)\phi(k)\frac{d^dk}{(2\pi)^d} \tag{1.34}$$

$$\begin{aligned}(1+2dl)(r+Audl) &= r' = r+dr\\ \Rightarrow \frac{dr}{dl} &= 2r+Au\end{aligned} \tag{1.35}$$

Similarly, at one-loop level we have

$$\begin{aligned}u' &= us^{4-d} = u(e^{l(4-d)}) \approx u(1+(4-d)dl)\\ \Rightarrow \frac{du}{dl} &= (4-d)u\end{aligned} \tag{1.36}$$

Here our final equations and $\beta-$functions are

$$\beta_r = \frac{dr}{dl} = 2r + Au \tag{1.37}$$

$$\beta_u = \frac{du}{dl} = (4-d)u \tag{1.38}$$

These flow equations have only one fixed point, the Gaussian fixed point is $K^* = (r = 0, u = 0)$.

The recursive relations can be linearized in the vicinity of the fixed point, by setting $r = r^* + \delta r$ and $u = u^* + \delta u$, as

$$\begin{pmatrix} \frac{d\delta r}{dl} \\ \frac{d\delta u}{dl} \end{pmatrix} = \begin{pmatrix} 2 & A \\ 0 & 4-d \end{pmatrix} \begin{pmatrix} \delta r \\ \delta u \end{pmatrix} \tag{1.39}$$

In the differential form of the recursion relations, the eigenvalues of the matrix determine the relevance of operators. This 2×2 matrix is not Hermitian. It has zero in the lower left side, so r does not generate any u. In contrast, u indeed generates r.

Because the lower-left element vanishes, the eigenvalues of the matrix is determined by the diagonal entries: $y_r = 2, y_u = 4 - d$. The asymmetic matrix has distinct left and right eigencectors. The right eigenvectors (that are our interests) are given by

$$|a\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |b\rangle = \begin{pmatrix} -\frac{A}{d-2} \\ 1 \end{pmatrix} \tag{1.40}$$

In terms of canonical operators, the eigenvectors correspond to

$$|a\rangle = 1 \times \phi^2 + 0 \times \phi^4 \tag{1.41}$$

$$|b\rangle = -\frac{A}{d-2} \times \phi^2 + 1 \times \phi^4 \tag{1.42}$$

Under the recursive RG operation, along the eigenvector direction we have $(\frac{dv_i}{d\ell} = y_i v_i \to v_i = e^{\ell y_i} = s^{y_i})$

$$T|a\rangle = s^2 |a\rangle \tag{1.43}$$

$$T|b\rangle = s^{4-d} |b\rangle \tag{1.44}$$

So we see parameter r is always relevant, which is associated with temperature in Landau phase transition theory. Actually we have another relevant parameter, h, the magnetic field. The parameter u is irrelevant for $d > 4$, while relevant for $d < 4$.

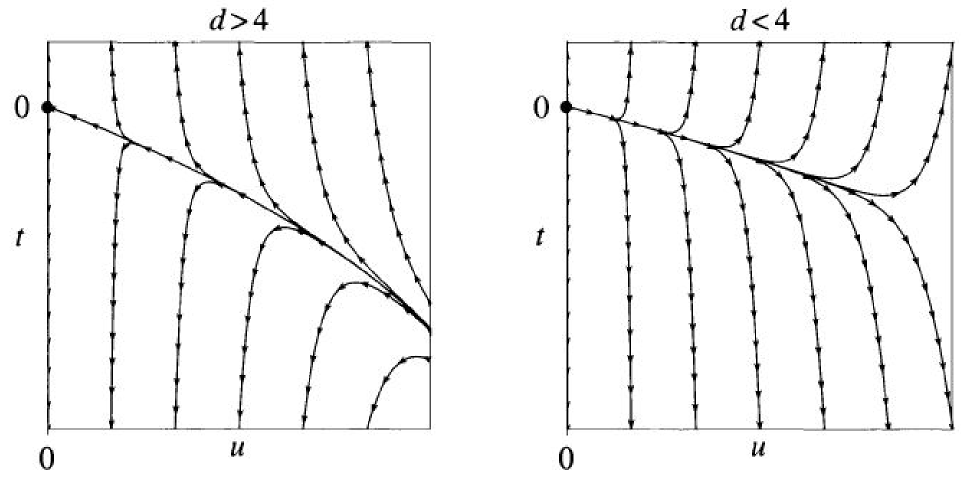

Figure 1.2: The RG flow diagram by renormalizing the quartic interaction with zero loop (at the first order).

Gaussian Model Exponents for $d > 4$

When $d > 4$, the parameter u is irrelevant. It is possible to tune one parameter to hit the critical space, see Fig. 1.2.

To find the exponents we need to begin with f, the free energy per unit volume. Please note that, the dropped term like $\ln Z_{fast}$ only contains the non-singlarity part. If we concern the critical exponents, they are unimportant. We only need to focus on Z_{slow}. Due to the change in scale that accompanies the RG, unit volume after RG corresponds to volume s^d before RG. Consequently, the free energy per unit volume behaves as follows in d dimesion:

$$f_s(r,u,h) = s^{-d} f_s(rs^2, us^{4-d}, hs^{1+d/2}) \tag{1.45}$$

Following the familiar route, we

$$\begin{aligned} & m(-r,u,h=0) = \frac{\partial f}{\partial h}|_{h=0} = s^{1-d/2} m(-rs^2, us^{4-d}, 0) \\ & = |r|^{-(1-d/2)/2} m(-1, ur^{(d-4)/2}, 0) \\ & \overset{u^*=0}{\Rightarrow} \beta = \frac{d-2}{4} \end{aligned} \tag{1.46}$$

Taking another h derivative,

$$\chi(-r,u,0) = s^{(1-d/2)+(1+d/2)} m(-rs^2, us^{4-d}, h=0) = |r|^1 m(-1,0,0), \ \to \gamma = 1 \tag{1.47}$$

Table 1.1: Gaussian model for $d > 4$ versus Landau theory.

Exponent	Landau	Gaussian $d > 4$
α	jump	$\frac{4-d}{2} < 0$
β	1/2	$\frac{d-2}{4}$
γ	1	1
δ	3	$\frac{d+2}{d-2}$
ν	1/2	1/2
η	0	0

To calculate specific heat, we take two derivatives of f with respect to r ($r \sim T - T_c = t$),

$$c_v = \frac{\partial^2 f}{\partial r^2}|_{r=0} = s^{4-d} f''(rs^2, 0, 0) = r^{-\frac{4-d}{2}} f(1, 0, 0), \ \Rightarrow \alpha = 2 - d/2 \tag{1.48}$$

Finally, to find δ we begin with f, take an $h-$derivative and then set $t = 0$:

$$m(0, u, h) = \frac{\partial f}{\partial h}|_{r=0} = s^{1-d/2} m(0, us^{4-d}, hs^{1+d/2}) = h^{\frac{d-2}{d+2}} m(0, uh^{(d-4)/(1+d/2)}, 1)$$
$$\Rightarrow h^{\frac{d-2}{d+2}} m(0, 0, 1), \ \rightarrow \delta = \frac{d+2}{d-2} \tag{1.49}$$

Now we compare the results from Gaussian model with Landau theory, in Tab. 1.3.1. The exponent β and δ agree only at $d = 4$.

Gaussian Model for $d < 4$

Both r and u are relevant, so we donot have stable fixed point.

1.3.2 Quartic term (Second order)

We will calculate the second order perturbations coming from quartic terms. We consider the Feynman diagram in Fig. 1.3. The one-loop correction to the coupling u is given by

$$\begin{aligned}
& -u^2[\int_{\Lambda/s}^{\Lambda} \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2+r)(|\mathbf{k}+\mathbf{k}_1-\mathbf{k}_3|^2+r)} \\
& + \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2+r)(|\mathbf{k}+\mathbf{k}_1-\mathbf{k}_4|^2+r)} \\
& + \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2+r)(|-\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2|^2+r)}] \\
& = -3u^2 \int_{\Lambda/s}^{\Lambda} \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2+r)^2} = -3u^2 A' dl
\end{aligned} \tag{1.50}$$

where we set all external momenta to 0 and we used

$$\begin{aligned}
& \int_{\Lambda/s}^{\Lambda} \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2+r)^2} = \int_{\Lambda/s}^{\Lambda} \frac{S_d k^{d-1} dk}{(2\pi)^d} \frac{1}{(k^2+r)^2} \\
& \approx \frac{S_d \Lambda^{d-2}}{(2\pi)^d} \int_{\Lambda/s}^{\Lambda} \frac{kdk}{(k^2+r)^2} \\
& = \frac{S_d \Lambda^{d-2}}{(2\pi)^d 2} [-\frac{1}{\Lambda^2+r} + \frac{1}{\Lambda^2/s^2+r}] \\
& \approx \frac{S_d \Lambda^{d-2}}{(2\pi)^d} \frac{2dl}{\Lambda^2} = \frac{S_d}{(2\pi)^d} \Lambda^{d-4} dl
\end{aligned} \tag{1.51}$$

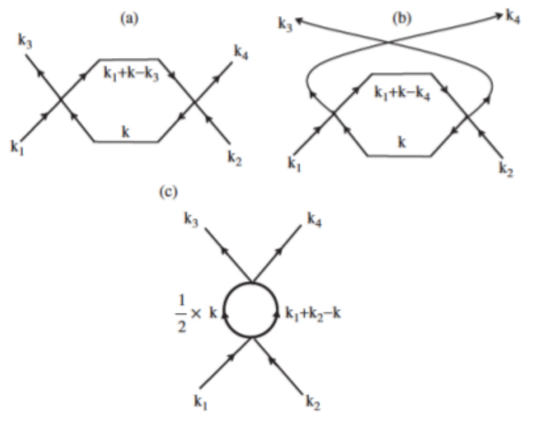

Figure 1.3: The Feynman diagrams that renormalize the quartic interaction at one loop.

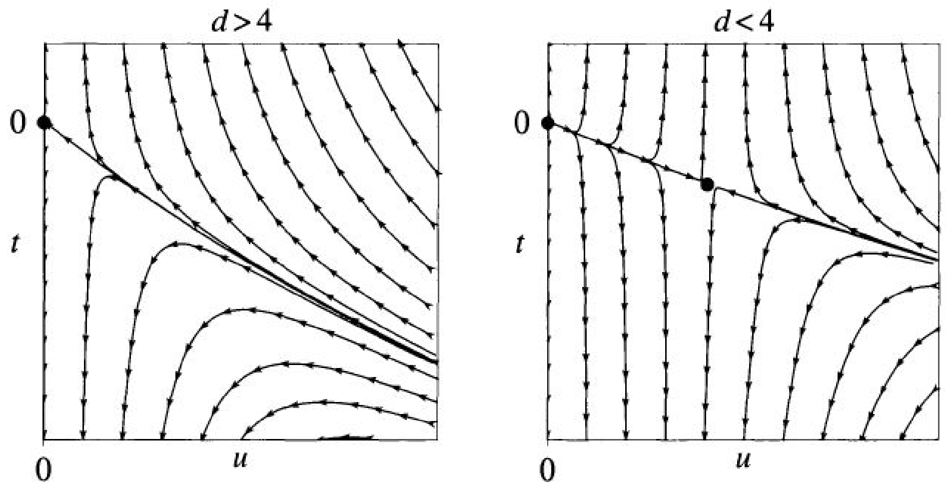

Figure 1.4: The RG flow diagram by renormalizing the quartic interaction at one loop.

Thus we rescale the momenta and field:

$$s^{4-d}\frac{u}{4}\int_{k<\Lambda}\prod_{i=1}^{3}\frac{d^dk'_i}{(2\pi)^d}\phi^*(k'_4)\phi^*(k'_3)\phi(k'_2)\phi(k'_1)\delta(4+3-2-1) \tag{1.52}$$

$$- s^{4-d}[3u^2A'dl]\int_{k<\Lambda}\prod_{i=1}^{3}\frac{d^dk'_i}{(2\pi)^d}\phi^*(k'_4)\phi^*(k'_3)\phi(k'_2)\phi(k'_1)\delta(4+3-2-1) \tag{1.53}$$

$$= s^{4-d}[u-3u^2A'dl]\int_{k<\Lambda}\prod_{i=1}^{3}\frac{d^dk'_i}{(2\pi)^d}\phi^*(k'_4)\phi^*(k'_3)\phi(k'_2)\phi(k'_1)\delta(4+3-2-1) \tag{1.54}$$

Then we conclude ($\varepsilon = 4-d$)

$$u' = (1+\varepsilon dl)[u-3u^2A'dl] \tag{1.55}$$

$$\frac{du}{dl} \approx \varepsilon u - 3u^2A' \tag{1.56}$$

The beta function is defined by

$$\beta_r = \frac{dr}{dl} = 2r + uA \tag{1.57}$$

$$\beta_u = \frac{du}{dl} = \varepsilon u - 3u^2A' \tag{1.58}$$

From this, we learn that we have two different fixed point: Gaussian $(r^*, u^*) = (0,0)$ and Wilson-Fisher (WF) $(r^*, u^*) = (-\frac{\varepsilon}{6}, \frac{\varepsilon}{3A})$ (we set $\Lambda = 1$).

The linearized flow near the WF fixed points is

$$\begin{pmatrix} \frac{d\delta r}{dl} \\ \frac{d\delta u}{dl} \end{pmatrix} = \begin{pmatrix} 2 & A \\ 0 & \varepsilon - 6Au^* \end{pmatrix} \begin{pmatrix} \delta r \\ \delta u \end{pmatrix} \tag{1.59}$$

Thus the eigenvalues are $y_r = 2, y_u = \varepsilon - 6Au^* = -\varepsilon$. $y_r > 0$ controls the instability of the fixed point. The second eigenvalue $y_u = \varepsilon > 0$ ($d < 4$) at the Gaussian point. Therefore, if $d < 4$, both directions become relevant at the Gaussian fixed point. (We have set $h = 0$) It is totally unstable. In contrast, $y_u = -\varepsilon$ becomes negative for $d < 4$. So one can tune parameters, namely r and u, to hit criticality. This does not correspond to any Ising-like transition. This fixed point will be of interest later on. This fixed point characterize the universality classes of rotational symmetry breaking in $d < 4$, with short-ranged interactions.

The divergence of the correlation length $\xi \sim t^\nu$ is controlled by the exponent y_r. And the other exponents can be computed in the similar way, using the new beta functions.

1.4 Continuous Symmetry

In the above sections, we discuss the statistical field theory for Ising model with Z_2 symmetry. As we have mentioned, the free energy could process different symmetries, which reflects the intrinsic properties of the different systems. For the systems with continuous $O(N)$ symmetry, we know the free energy is like

$$f = f_0 + a(T)\mathbf{m}^2 + b(T)\mathbf{m}^4 - h \cdot \mathbf{m} + ... \tag{1.60}$$

where $\mathbf{m}_i$ is a vector but not a number. Similarly, when going to the field theory, the free-energy functional changes to

$$Z = \int D[\phi] \exp[-\int d^d x(-\nabla\vec{\phi})^2 + r|\vec{\phi}|^2 + u|\vec{\phi}|^4)] \tag{1.61}$$

where $\vec{\phi}(x) = (\phi_1(x), \phi_2(x), ...)$. Similarly, we can discuss the RG of this effective field with continuous symmetry. For superfluid transition (in ^{4}He), there are two real components ($N = 2$). For $N = 3$, the symmetry is belong to Heisenberg model. Next we discuss an action that enjoys a $O(2)$ (or $U(1)$ symmetry: $S[\phi, \phi^*] = S[\phi e^{i\theta}, \phi^* e^{-i\theta}]$ where θ is arbitrary. (We can also see this as $O(2)$ symmetry of S under rotations of the real and imaginary parts of ϕ.)

1.4.1 Gaussian Fixed Point

Let us recall the effective ϕ-4 theory theory of Ising model:

$$S[\phi] = \int d^d x((-\nabla\phi)^2 + r\phi^2 + u\phi^4) \tag{1.62}$$

When $c_4 = 0$, this is called Gaussian model and it is solvable. Please note that, here $\phi(x)$ is a field describing the physics around the critical point. Instead of removing fluctuations at scales $a < x < sa$, we now remove Fourier modes in the momentum shell $\Lambda/s < q < \Lambda$ to implement the coarse-graining.

The partition function of the Gaussian model (in momentum space) is

$$Z = \int [D\phi(k)D\phi^*(k)]e^{-S[\phi,\phi^*]} = \int [D\phi(k)\phi^*(k)] \exp[-\int_0^\Lambda \frac{d^d k}{(2\pi)^d}\phi^*(\mathbf{k})k^2\phi(\mathbf{k})] \tag{1.63}$$

We will study $\Lambda \sim 1/a$ centered at the origin of BZ and ignore the shape of the BZ.

The dimension d is usually an integer, but we mush be prepared to work with continuous d. For our discussions, we need the integration measure for rotationally invariant integrands:

$$d^d k = k^{d-1} dk S_d, \quad S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)} \tag{1.64}$$

At the first step, we divide the field into slow and fast modes:

$$\phi(k) = \begin{cases} \phi_s(\mathbf{k}), 0 < |\mathbf{k}| < \Lambda/s \\ \phi_f(\mathbf{k}), \Lambda/s < |\mathbf{k}| < \Lambda \end{cases} \tag{1.65}$$

where $s > 1$ is the scale parameter.

The action itself becomes

$$S[\phi, \phi^*] = S_{\text{slow}} + S_{\text{fast}} = \int_0^{\Lambda/s} \frac{d^d k}{(2\pi)^d} \phi_s^*(\mathbf{k}) k^2 \phi_s(\mathbf{k}) + \int_{\Lambda/s}^{\Lambda} \frac{d^d k}{(2\pi)^d} \phi_f^*(\mathbf{k}) k^2 \phi_f(\mathbf{k}) \tag{1.66}$$

Thus integrating over the fast modes just gives an overall constant multiplying the Z for the slow modes:

$$\begin{aligned} Z &= \int [D\phi_s(k)] e^{-S[\phi_s]} \int [D\phi_f(k)] e^{-S[\phi_f]} \\ &= \int [D\phi_s(k)] e^{-S[\phi_s] + \ln Z_{fast}} \end{aligned} \tag{1.67}$$

$$Z_{fast} = \int [D\phi_f(k)] e^{-S[\phi_f]} \tag{1.68}$$

We ignore Z_{fast} because it is independent of ϕ_s.

At the second step, the action after mode elimination followed by a new momentum $k' = sk$, then

$$\begin{aligned} S[\phi] &= \int_0^{\Lambda/s} \frac{d^d k}{(2\pi)^d} \phi_s^*(k) k^2 \phi_s(k) \\ &= \int_0^{\Lambda} \frac{d^d k'}{s^d (2\pi)^d} \phi_s^*(k'/s) k'^2/s^2 \phi_s(k'/s) \\ &= s^{-(d+2)} \int_0^{\Lambda} \frac{d^d k'}{(2\pi)^d} \phi_s^*(k'/s)^2 k^2 \phi_s(k'/s) \end{aligned} \tag{1.69}$$

At the third step, to take care of the factor $s^{-(d+2)}$ we introduce a rescaled field

$$\phi'(k') = s^{-(d/2+1)}\phi_s(k'/s) \tag{1.70}$$

and we get

$$S[\phi'] = \int_0^\Lambda \frac{d^d k'}{(2\pi)^d}(\phi'(k'))^* k'^2 \phi'(k') \tag{1.71}$$

With this definition of the RG transformation, we have a mapping from actions defined in a certain k-space to actions in the same space. Thus, if we represent the initial action as a point in a coupling constant space, this point will flow under the RG transformation to another point in the same space. If we make a connection with the real-space RG, the first step is make a bulk spin, the second step is to do the summation and to construct the effective model, and the third step is to rescale (renormalize) the field.

We now want to determine the flow of couplings near this fixed point. We will do this by adding some perturbations in linear, quadratic and quartic in ϕ.

linear operator

The uniform magnetic field couples linearly to ϕ as $h\phi(k=0) + h^*\phi^*(k=0)$. When defining the rescaled field, we see

$$h\phi(0) = hs^{(d/2+1)}\phi'(0\cdot s) = h_s\phi'(0), \quad h_s = hs^{1+d/2} \tag{1.72}$$

Here we see an infinitesimal h gets amplified by the RG process, so h is relevant parameter.

Comparing it with the scaling hyphethesis, we see $y_h = 1 + d/2$ for Gaussian field theory.

quadratic operator

$$\begin{aligned}
S_r &= \int_0^\Lambda \frac{d^dk}{(2\pi)^d}\phi^*(\mathbf{k})r\phi(\mathbf{k}) \\
&= const. \int_0^{\Lambda/s} \frac{d^dk}{(2\pi)^d}\phi_s^*(\mathbf{k})r\phi_s(\mathbf{k}) = const. \int_0^\Lambda \frac{d^dk'}{s^d(2\pi)^d}\phi_s^*(\mathbf{k}'/s)r\phi_s(\mathbf{k}'/s) \\
&= s^2 \int_0^\Lambda \frac{d^dk}{(2\pi)^d}\phi'^*(\mathbf{k}')r\phi(\mathbf{k}') \equiv \int_0^\Lambda \frac{d^dk}{(2\pi)^d}\phi'^*(\mathbf{k}')r_s\phi(\mathbf{k}') \\
&\Rightarrow r_s = rs^2
\end{aligned} \tag{1.73}$$

Again, we have identified another relevant parameter in r.

Comparing the scaling of $r_s = rs^2$ with the scaling hyphothesis, we can infer $y_t = 2$ (please note that $r \sim a(T) \propto (T - T_c) = t$). So we may identify r with t, the dimensionless temperature that takes us off criticality.

Since the correlation length ξ drops by a factor of $1/s$ under this rescaling of momentum by s, we have

$$\begin{aligned}
&\xi(r)/s = \xi(r_s) \\
&\Rightarrow \xi(r) = s\xi(rs^2) = r^{-\frac{1}{2}}\xi(1) \Rightarrow \nu = \frac{1}{2}
\end{aligned} \tag{1.74}$$

Thus, we have $y_t = 2$ and $y_h = 1 + d/2$, $\nu = 1/2$, from which we can calculate the critical exponent

$$\alpha = \frac{4-d}{2}, \beta = \frac{d-2}{4}, \gamma = 1, \delta = \frac{d+2}{d-2}, \nu = \frac{1}{2}, \eta = 0 \tag{1.75}$$

1.4.2 Wilson-Fisher critical point

Let us consider a perturbation around the Gaussian point. This will lead to a new fixed point. When we consider quartic perturbations of the fixed point, we run into a new complication: the term couples slow and fast modes and we have to do more than just rewrite the old perturbation in terms of new fields and new momenta.

Quartic operator (First order)

$$S = S_0 + S_u = \int_0^\Lambda \frac{d^d k}{(2\pi)^d} \phi^*(\mathbf{k})(k^2 + r)\phi(\mathbf{k}) + \frac{u}{4}\int_{|k|<\Lambda} \frac{d^d k_{i=1,2,3,4}}{(2\pi)^d} \phi^*(\mathbf{k}_4)\phi^*(\mathbf{k}_3)\phi(\mathbf{k}_2)\phi(\mathbf{k}_1) \tag{1.76}$$

Integrating over fast modes:

$$\begin{aligned} Z &= \int [D\phi_s] e^{-S_0[\phi_s]} \int [D\phi_f] e^{-S_0(\phi_f)} e^{-S_u[\phi_s,\phi_f]} \\ &= \int [D\phi_s] e^{-S_{eff}[\phi_s]} \end{aligned} \tag{1.77}$$

The effective action is

$$\begin{aligned} e^{-S_{eff}[\phi_s]} &= e^{-S_0[\phi_s]} \int [D\phi_f] e^{-S_0(\phi_f)} e^{-S_u[\phi_s,\phi_f]} \\ &= e^{-S_0[\phi_s]} \frac{\int [D\phi_f] e^{-S_0(\phi_f)} e^{-S_u[\phi_s,\phi_f]}}{\int [D\phi_f] e^{-S_0[\phi_f]}} \int [D\phi_f] e^{-S_0[\phi_f]} \\ &= e^{-S_0[\phi_s]} \langle e^{-S_u[\phi_s,\phi_f]} \rangle_f \\ &= e^{-[S_0[\phi_s] + \langle S_u[\phi_s,\phi_f]\rangle + \frac{1}{2}[\langle S_u[\phi_s,\phi_f]^2\rangle - \langle S_u[\phi_s,\phi_f]\rangle^2 + \ldots]} \end{aligned} \tag{1.78}$$

Here we utilize the cumulant expansion, which relates the mean of the exponential to the exponential of the means,

$$\langle e^{\Omega} \rangle = e^{\langle\Omega\rangle + \frac{1}{2}[\langle\Omega^2\rangle - \langle\Omega\rangle^2 + \ldots]} \tag{1.79}$$

The leading term in the cumulant expansion has the form

$$\langle S_u \rangle = \frac{u}{4} \langle \int_{k<\Lambda} (\phi_f + \phi_s)^*_4 (\phi_f + \phi_s)^*_3 (\phi_f + \phi_s)_2 (\phi_f + \phi_s)_1 \rangle \tag{1.80}$$

There are 16 possibilities falling into four groups: 8 terms with an odd number of fast modes, 1 term with all fast modes, 1 term with all slow modes, and 6 terms with two slow and two fast modes. The terms with odd number of fast modes donot contribute the action, we ignore them. In the six terms of two fast modes, the term $\phi^*_f \phi^*_f \phi_s \phi_s$, $\phi^*_s \phi^*_s \phi_f \phi_f$ should vanish, and we need consider the terms $\phi_f^*\phi_s^*\phi_s\phi_f$, $\phi_s^*\phi_f^*\phi_f\phi_s$, $\phi_s^*\phi_f^*\phi_s\phi_f$, $\phi_f^*\phi_s^*\phi_f\phi_s$. So we have

$$\begin{aligned}\langle S_u\rangle &= u\int \phi_s^*(k_4)\phi_s(k_2)\langle\phi_f^*(k_3)\phi_f(k_1)\rangle(2\pi)^d\delta(k_4+k_3-k_2-k_1)\frac{d^dk}{(2\pi)^d}\\ &= \int_0^{\Lambda/s}\phi_s^*(k)\phi_s(k)\frac{d^dk}{(2\pi)^d}[u\int_{\Lambda/s}^{\Lambda}\frac{d^dk_3}{(2\pi)^d}\frac{1}{k_3^2+r}]\\ &= \int_0^{\Lambda/s}\phi_s^*(k)\phi_s(k)\frac{d^dk}{(2\pi)^d}[u\frac{S_d}{(2\pi)^d}\frac{\Lambda^d}{\Lambda^2+r}dl],\end{aligned}\tag{1.81}$$

where we used $s=e^l\approx 1+dl$ and $dl\ll 1$. The integral needs special treatment

$$\begin{aligned}&\int_{\Lambda/s}^{\Lambda}\frac{d^dk}{(2\pi)^d}\frac{1}{k^2+r}=\int_{\Lambda/s}^{\Lambda}\frac{S_dk^{d-1}dk}{(2\pi)^d}\frac{1}{k^2+r}\\ &\approx\frac{S_d\Lambda^{d-2}}{(2\pi)^d}\int_{\Lambda/s}^{\Lambda}\frac{kdk}{k^2+r}\\ &=\frac{S_d\Lambda^{d-2}}{(2\pi)^d2}[\ln(\Lambda^2+r)-\ln(\Lambda^2/s^2+r)]\\ &\approx\frac{S_d\Lambda^{d-2}}{(2\pi)^d2}[\ln(\Lambda^2+r)-\ln(\Lambda^2(1-2dl)+r)]\\ &=\frac{S_d\Lambda^{d-2}}{(2\pi)^d2}[\ln(\Lambda^2+r)-\ln[(\Lambda^2+r)(1-\frac{2dl\Lambda^2}{\Lambda^2+r})]]\\ &\approx\frac{S_d\Lambda^{d-2}}{(2\pi)^d2}\frac{2dl\Lambda^2}{\Lambda^2+r}=\frac{S_d}{(2\pi)^d}\frac{\Lambda^d}{\Lambda^2+r}dl\end{aligned}\tag{1.82}$$

We then make the rescaling

$$k'=sk,\quad \phi'(k')=s^{-(d/2+1)}\phi(k'/s)\tag{1.83}$$

we have

$$\langle S_u\rangle = s^2\int_0^{\Lambda}\phi^*(k')\phi(k')\frac{d^dk'}{(2\pi)^d}[uAdl]+s^{4-d}\frac{u}{4}\int_{k<\Lambda}\prod_{i=1}^{3}\frac{d^dk_i'}{(2\pi)^d}\phi^*(k_4')\phi^*(k_3')\phi(k_2')\phi(k_1')\delta(4+3-2-1)\tag{1.84}$$

We obtain the term like

$$(1+2dl)\int_0^{\Lambda}\phi^*(k)\phi(k)(r+Audl)\frac{d^dk}{(2\pi)^d}=\int_0^{\Lambda}r'\phi^*(k)\phi(k)\frac{d^dk}{(2\pi)^d}\tag{1.85}$$

$$(1+2dl)(r+Audl) = r' = r+dr$$
$$\Rightarrow \frac{dr}{dl} = 2r + Au \tag{1.86}$$

Similarly, at one-loop level we have

$$u' = us^{4-d} = u(e^{l(4-d)}) \approx u(1+(4-d)dl)$$
$$\Rightarrow \frac{du}{dl} = (4-d)u \tag{1.87}$$

Here our final equations and $\beta-$functions are

$$\beta_r = \frac{dr}{dl} = 2r + Au \tag{1.88}$$
$$\beta_u = \frac{du}{dl} = (4-d)u \tag{1.89}$$

These flow equations have only one fixed point, the Gaussian fixed point is $K^* = (r=0, u=0)$.

The recursive relations can be linearized in the vicinity of the fixed point, by setting $r = r^* + \delta r$ and $u = u^* + \delta u$, as

$$\begin{pmatrix} \frac{d\delta r}{dl} \\ \frac{d\delta u}{dl} \end{pmatrix} = \begin{pmatrix} 2 & A \\ 0 & 4-d \end{pmatrix} \begin{pmatrix} \delta r \\ \delta u \end{pmatrix} \tag{1.90}$$

In the differential form of the recursion relations, the eigenvalues of the matrix determine the relevance of operators. This 2×2 matrix is not Hermitian. It has zero in the lower left side, so r does not generate any u. In contrast, u indeed generates r.

Because the lower-left element vanishes, the eigenvalues of the matrix is determined by the diagonal entries: $y_r = 2, y_u = 4-d$. The asymmetic matrix has distinct left and right eigencectors. The right eigenvectors (that are our interests) are given by

$$|a\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |b\rangle = \begin{pmatrix} -\frac{A}{d-2} \\ 1 \end{pmatrix} \tag{1.91}$$

In terms of canonical operators, the eigenvectors correspond to

$$|a\rangle = 1 \times \phi^2 + 0 \times \phi^4 \tag{1.92}$$

$$|b\rangle = -\frac{A}{d-2} \times \phi^2 + 1 \times \phi^4 \tag{1.93}$$

Under the recursive RG operation, along the eigenvector direction we have ($\frac{dv_i}{d\ell} = y_i v_i \rightarrow v_i = e^{\ell y_i} = s^{y_i}$)

$$T|a\rangle = s^2|a\rangle \tag{1.94}$$

$$T|b\rangle = s^{4-d}|b\rangle \tag{1.95}$$

So we see parameter r is always relevant, which is associated with temperature in Landau phase transition theory. Actually we have another relevant parameter, h, the magnetic field. The parameter u is irrelevant for $d > 4$, while relevant for $d < 4$.

The discussion of RG flow is similar to the previous discussion, because the flow equation are the same.

Quartic term (Second order)

We will calculate the second order perturbations coming from quartic terms. We consider the Feynman diagram in Fig. 1.3. The one-loop correction to the coupling u is given by

$$\begin{aligned}
&-u^2[\int_{\Lambda/s}^{\Lambda} \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2+r)(|\mathbf{k}+\mathbf{k}_1-\mathbf{k}_3|^2+r)} \\
&+ \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2+r)(|\mathbf{k}+\mathbf{k}_1-\mathbf{k}_4|^2+r)} \\
&+ \frac{1}{2}\frac{d^dk}{(2\pi)^d} \frac{1}{(k^2+r)(|-\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2|^2+r)}] \\
&= -\frac{5u^2}{2}\int_{\Lambda/s}^{\Lambda} \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2+r)^2} = -\frac{5u^2 A' dl}{2}
\end{aligned} \tag{1.96}$$

where we set all external momenta to 0 and we used

$$\begin{aligned}&\int_{\Lambda/s}^{\Lambda}\frac{d^dk}{(2\pi)^d}\frac{1}{(k^2+r)^2}=\int_{\Lambda/s}^{\Lambda}\frac{S_dk^{d-1}dk}{(2\pi)^d}\frac{1}{(k^2+r)^2}\\&\approx\frac{S_d\Lambda^{d-2}}{(2\pi)^d}\int_{\Lambda/s}^{\Lambda}\frac{kdk}{(k^2+r)^2}\\&=\frac{S_d\Lambda^{d-2}}{(2\pi)^d2}[-\frac{1}{\Lambda^2+r}+\frac{1}{\Lambda^2/s^2+r}]\\&\approx\frac{S_d\Lambda^{d-2}}{(2\pi)^d}\frac{2dl}{\Lambda^2}=\frac{S_d}{(2\pi)^d}\Lambda^{d-4}dl\end{aligned}\tag{1.97}$$

Thus we rescale the momenta and field:

$$s^{4-d}\frac{u}{4}\int_{k<\Lambda}\prod_{i=1}^{3}\frac{d^dk_i'}{(2\pi)^d}\phi^*(k_4')\phi^*(k_3')\phi(k_2')\phi(k_1')\delta(4+3-2-1)\tag{1.98}$$

$$-s^{4-d}[\frac{5u^2A'dl}{2}]\int_{k<\Lambda}\prod_{i=1}^{3}\frac{d^dk_i'}{(2\pi)^d}\phi^*(k_4')\phi^*(k_3')\phi(k_2')\phi(k_1')\delta(4+3-2-1)\tag{1.99}$$

$$=s^{4-d}[u-\frac{5u^2A'dl}{2}]\int_{k<\Lambda}\prod_{i=1}^{3}\frac{d^dk_i'}{(2\pi)^d}\phi^*(k_4')\phi^*(k_3')\phi(k_2')\phi(k_1')\delta(4+3-2-1)\tag{1.100}$$

Then we conclude ($\varepsilon=4-d$)

$$u'=(1+\varepsilon dl)[u-\frac{5u^2A'dl}{2}]\tag{1.101}$$

$$\frac{du}{dl}\approx\varepsilon u-\frac{5u^2A'}{2}\tag{1.102}$$

The beta function is defined by

$$\beta_r=\frac{dr}{dl}=2r+uA\tag{1.103}$$

$$\beta_u=\frac{du}{dl}=\varepsilon u-\frac{5u^2A'}{2}\tag{1.104}$$

From this, we learn that we have two different fixed point: Gaussian $(r^*,u^*)=(0,0)$ and Wilson-Fisher (WF) $(r^*,u^*)=(-\frac{\varepsilon}{5},\frac{2\varepsilon}{5A})$ (we set $\Lambda=1$).

The linearized flow near the WF fixed points is

$$\begin{pmatrix}\frac{d\delta r}{dl}\\ \frac{d\delta u}{dl}\end{pmatrix}=\begin{pmatrix}2 & A\\ 0 & \varepsilon-5Au^*\end{pmatrix}\begin{pmatrix}\delta r\\ \delta u\end{pmatrix}\tag{1.105}$$

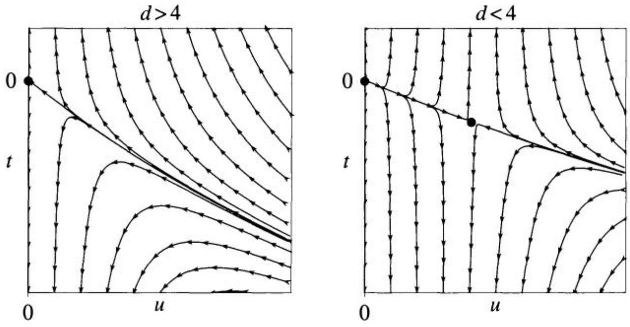

Figure 1.5: The RG diagrams that renormalize the quartic interaction at one loop.

Thus the eigenvalues are $y_r = 2, y_u = \varepsilon - 5Au^* = -\varepsilon$. $y_r > 0$ controls the instability of the fixed point. The second eigenvalue $y_u = \varepsilon > 0$ $(d < 4)$ at the Gaussian point. Therefore, if $d < 4$, both directions become relevant at the Gaussian fixed point. (We have set $h = 0$) It is totally unstable. In contrast, $y_u = -\varepsilon$ becomes negative for $d < 4$. So one can tune parameters, namely r and u, to hit criticality. This does not correspond to any Ising-like transition. This fixed point will be of interest later on. This fixed point characterize the universality classes of rotational symmetry breaking in $d < 4$, with short-ranged interactions.

The divergence of the correlation length $\xi \sim t^{\nu}$ is controlled by the exponent y_r. And the other exponents can be computed in the similar way, using the new beta functions.

1.5 Dimension analysis

The quantity of free energy $F[\phi]$ should be dimensionless.

The relevance of scaling is shown below. Consider an interaction term:

$$F[\phi] \sim \int d^d x g_O \hat{O}(x) \tag{1.106}$$

Here $\vec{O}$ can be any form such as $\phi^n, \phi^m(\nabla\phi)^2$. Under the scaling $x \to xs$, the operator is rescaled as

$$\hat{O}(x) \to \hat{O}'(x') = s^{\Delta_O} O(x) \tag{1.107}$$

from the free energy functional, we have the scaling dimension of the coupling is

$$\Delta_{g_O} = d - \Delta_O \tag{1.108}$$

That is, $g_O \to g_O s^{d-\Delta_O}$. We can see immediately that g_O either diverges or vanishes as we push forwards with the RG: 1) $\Delta_O < d$: relevant; 2) $\Delta_O > d$: irrelevant; 3) $\Delta_O = d$: marginal.

For the interaction term $g_n\phi^n$ in the phi-4 theory, under the RG

$$g_n\phi^n \to s^d g_n s^{-n\Delta_\phi}(\phi')^n \tag{1.109}$$

To restore the invariance of quadratic term, we have $\Delta_\phi = \frac{d-2}{2}$, then

$$g_n(s) = g_n s^{d-n\Delta_\phi} = s^{(1-n/2)d+n} g_n \tag{1.110}$$

So we see that, for example, ϕ^4 is irrelevant for $d > 4$ while relevant for $d < 4$, and marginal for $d = 4$. ϕ^6 is irrelevant in $d > 3$, relevant in $d < 3$, marginal in $d = 3$.