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# Chapter 1

# Phonons and interaction with Electrons

In this chapter we turn to phonons and their interactions with electrons. These interactions play an important role in condensed matter physics. For example, at room temperature, the resistivity of metals results mainly from electron-phonon interaction. At low temperature, the electron-phonon interaction is responsible for the superconducting properties of many metals. We will discuss the electron-phonon interaction in detail. At last, we provide a microscopic derivation of Ginzburg-Landau theory for superconductivity.



## 1.1 Basics

### 1.1.1 Lattice vibrations

The simplest case we can deal with is a one-dimensional crystal with one atom per unit cell. Consider a line of $N$ atoms, each of mass $M$. In equilibrium, the position of atom n is $R_n = na$, and the separation between adjacent atoms is $a$. We model the interatomic interactions by massless springs, each of force constant $k$, which connect neighboring atoms. When atoms vibrate, they are displaced from equilibrium. Let $u_n$ be the displacement from equilibrium of atom n. We adopt periodic boundary conditions: $u_n = u_{N+1}$. Newton 's second law gives

$$M\,\ddot{u_n} = k(u_{n-1} - 2u_n + u_{n+1}) \tag{1.1}$$

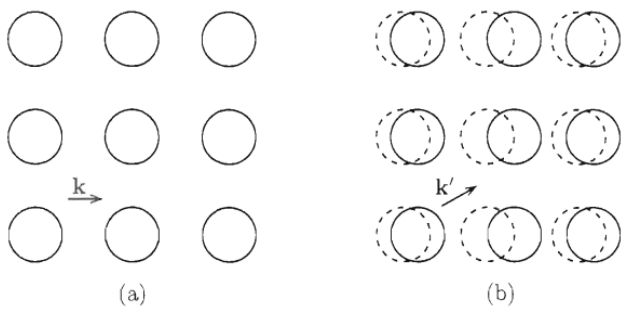



Figure 1.1: An electron is moving in a static ion potential, versus in the presence of phonon, the ions lattice vibrate, and electron sees a different potentials.

In a normal mode all atoms vibrate with the same wave vector and frequency, denoting wave vector as $q$ and frequency as $\omega_q$. The atom n in a normal mode has a displacement given by

$$u_n = A\exp[i(qR_n - \omega_q t)] \rightarrow -M\omega_q^2 = 2k(\cos(qa) - 1) = 4k\sin^2(qa/2) \tag{1.2}$$

The relation between $\omega_q$ and q is known as the dispersion relation.

Next task is to construct an expression for the energy of the atoms in terms of the normal coordinates $u_n = \frac{1}{\sqrt{NM}}\sum_q Q_q e^{iqR_n}$. The kinetic energy is given by

$$T = \frac{M}{2}\sum_{n=1}^{N}(\dot{u_n})^2 = \frac{1}{2N}\sum_n\sum_{q,q'}\dot{Q_q}\,\dot{Q_{q'}}\,e^{i(q+q')R_n} = \frac{1}{2}\sum_q \dot{Q_q}\,\dot{Q_{-q}} \tag{1.3}$$

The potential energy is the elastic energy of the springs

$$V = \frac{k}{2}\sum_n (u_{n+1} - u_n)^2 = \frac{1}{2}\sum_q \omega_q^2 Q_q Q_{-q} \tag{1.4}$$

where we used the relation $\sum_n (u_{n+1}-u_n)^2 = \sum_n \frac{1}{NM}[\sum_q Q_q e^{iqR_n}(e^{iqa}-1)]^2 = \sum_n \frac{1}{NM}\sum_{q,q'}[Q_q Q_{q'} e^{i(q+q')R_n}($
$1)(e^{iq'a}-1)] = \sum_q Q_q Q_{-q}|e^{iqa}-1|^2 = \sum_q Q_q Q_{-q}\sin^2(qa/2)$.

The Lagrangian $L = T - V$ is thus a function of the normal coordinates. The canonical momentum conjugate to $Q_q$ is

$$P_q = \frac{\partial L}{\partial \dot{Q_q}} = \dot{Q_{-q}} \tag{1.5}$$

The Hamiltonian is expressed as

$$H = \frac{1}{2}\sum_q (P_q P_{-q} + \omega_q^2 Q_q Q_{-q}) \tag{1.6}$$

In first quantization scheme, the above operators satisfy the commutation relations

$$[Q_q, Q_{q'}] = [P_q, P_{q'}] = 0, [Q_q, P_{q'}] = i\hbar\delta_{q,q'} \tag{1.7}$$

Then we define the new operators,

$$a_q = (2\hbar\omega)^{-1/2}(\omega_q Q_q + iP_{-q}), a_q^\dagger = (2\hbar\omega)^{-1/2}(\omega_q Q_{-q} - iP_q) \tag{1.8}$$

$$[a_q, q_{q'}] = [a_q^\dagger, a_{q'}^\dagger] = 0, [a_q, a_{q'}^\dagger] = \delta_{q,q'} \tag{1.9}$$

$$\Rightarrow H = \sum_q \hbar\omega_q (a_q^\dagger a_q + \frac{1}{2}) \tag{1.10}$$

The Hamiltonian is seen to be a collection of N independent harmonic oscillators. The eigenvalues are $\sum_q \hbar\omega_q(n_q + 1/2)$. $n_q$ particle-like excitation is phonon, which is quantized of vibrations of lattice. Since they satify the commutation relation, phonon is boson. Each phonon of wave number $q \neq 0$ represents a traveling wave of wavelength $2\pi/q$.

*One-dimensional diatomic lattice: optical phonon*

Phonons are the quanta of lattice oscillations. Phonons can be either acoustical or optical. The difference is that the frequency of acoustical phonons goes to zero as the wave vector $q \to 0$, while for optical phonons it stays finite. Acoustical phonons are present in any crystal, representing, in fact, Goldstone mode related to broken translation symmetry. Optical phonons appear only in crystals with more than one atom in elementary cell.

### 1.1.2 Electron-phonon interaction

The basic idea underlying the electron-phonon interaction is simple, as illustrated in Figure 1.1. When ions sit at their equilibrium positions, the state of an electron is described by a Bloch function of wave vector **k** (and spin projection a and band index n). A phonon disturbs the lattice, and ions move out of their equilibrium positions. This causes a change in the potential

seen by the electron (the potential no longer has the periodicity of the lattice). This change, in tum, scatters the electron into another state with wave vector $\mathbf{k}'$.

In this section, we calculate the electron-phonon interaction within the rigid-ion approximation. For simplicity of notation, we assume that there is one atom per unit cell; the extension to a crystal with a basis is straightforward. The interaction of an electron at position $\mathbf{r}_j$ with the ions is given by

$$V_{lattice} = \sum_n V(\mathbf{r}_j - \mathbf{R}_n - \mathbf{u}_n) \approx \sum_n V(\mathbf{r}_j - \mathbf{R}_n) - \sum_n \mathbf{u}_n \cdot \nabla V(\mathbf{r}_j - \mathbf{R}_n) \tag{1.11}$$

So the electron-phonon interaction is

$$\begin{aligned} V_{e-p} &= -\sum_n \mathbf{u}_n \cdot \nabla V(\mathbf{r}_j - \mathbf{R}_n) \\ &= \frac{-1}{\sqrt{NM}} \sum_n \sum_{q\lambda} Q_{q\lambda} e^{iqR_n} \epsilon_\lambda \nabla V(\mathbf{r}_j - \mathbf{R}_n) \\ &= -\sqrt{N/M} \sum_k \sum_{q\lambda,G} \epsilon_\lambda T(k,q,G) c^\dagger_{k+q+G} c_k Q_{q\lambda} \end{aligned} \tag{1.12}$$

where we used

$$\begin{aligned} &\epsilon_\lambda \nabla V(\mathbf{r} - \mathbf{R}_n) \to \sum_{k,k'} \langle k' | \epsilon_\lambda \nabla V(\mathbf{r} - \mathbf{R}_n) | k \rangle c^\dagger_{k'} c_k \\ &= \sum_{k,k'} [\int d\mathbf{r} \psi^*_{k'}(\mathbf{r}) \epsilon_\lambda \nabla V(\mathbf{r} - \mathbf{R}_n) \psi_k(\mathbf{r})] c^\dagger_{k'} c_k \\ &= \sum_{k,k'} e^{i(k-k')R_n} [\int d\mathbf{r} \psi^*_{k'}(\mathbf{r}) \epsilon_\lambda \nabla V(\mathbf{r}) \psi_k(\mathbf{r})] c^\dagger_{k'} c_k \end{aligned} \tag{1.13}$$

$$\sum_n e^{i(k-k'+q)R_n} = N \sum_G \delta_{k',k+q+G} \tag{1.14}$$

$$\begin{aligned} T(k,q,G) &= \int d\mathbf{r} \psi^*_{k+q+G}(\mathbf{r}) \nabla V(\mathbf{r}) \psi_k(\mathbf{r}) \\ &\sim \int dr e^{-i\mathbf{k}\cdot\mathbf{r}} (-i \sum_p p V_p e^{ipr}) e^{i\mathbf{r}(k+q+G)\cdot\mathbf{r}} \\ &\sim -i V_{q+G}((\mathbf{q}+\mathbf{G})) \end{aligned} \tag{1.15}$$

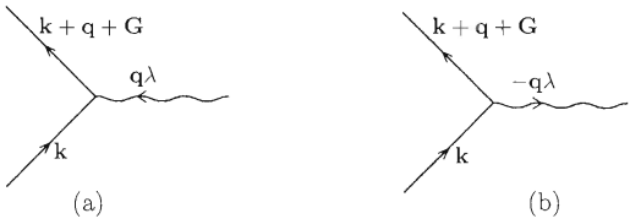



Figure 1.2: Feynman diagram for electron-phonon coupling.

We finally obtain

$$V_{e-ph} = -\frac{i}{V}\sum_{k,q,G}\sqrt{\frac{N\hbar}{2M\omega_{q\lambda}}}V_{q+G}(q+G)\epsilon_\lambda c^\dagger_{k+q+G}c_k(a_{q\lambda}+a^\dagger_{-q\lambda}) \tag{1.16}$$

$$= \sum_{k,q,G} g_{q\lambda}c^\dagger_{k+q}c_k(a_{q\lambda}+a^\dagger_{-q\lambda}), g_{q\lambda} = \frac{i}{V}\sqrt{\frac{N\hbar}{2M\omega_q}}\frac{4\pi Ze^2}{q^{TF}}\vec{\epsilon}_\lambda\cdot\vec{q} \tag{1.17}$$

We make the following remarks regarding the electron-phonon interaction. The general Feymann diagram is shown in Fig. 1.2.

### 1.1.3 Phonon Green's function

The phonon retarded green's function is defined by

$$D^R(\mathbf{q}\lambda,t) = -i\Theta(t)\langle[\phi_{\mathbf{q},\lambda}(t),\phi^\dagger_{\mathbf{q},\lambda}(0)]\rangle, \phi_{\mathbf{q},\lambda} = a_{\mathbf{q},\lambda}+a^\dagger_{-\mathbf{q},\lambda} \tag{1.18}$$

The phonon imaginary time Green's function is

$$D(\mathbf{q}\lambda,\tau) = -\langle T_\tau\phi_{\mathbf{q},\lambda}(\tau)\phi^\dagger_{\mathbf{q},\lambda}(0)\rangle, T_\tau\phi_{\mathbf{q},\lambda}(\tau)\phi^\dagger_{\mathbf{q},\lambda}(0) = \begin{cases}\phi_{\mathbf{q},\lambda}(\tau)\phi^\dagger_{\mathbf{q},\lambda}(0), \tau>0\\ \phi^\dagger_{\mathbf{q},\lambda}(0)\phi_{\mathbf{q},\lambda}(\tau), \tau<0\end{cases} \tag{1.19}$$

The time $\tau$ is restricted to the interval $[-\beta\hbar,\beta\hbar]$, and it gives the period for $D(\mathbf{q}\lambda,\tau)$:

$$D(\mathbf{q}\lambda,\tau) = \frac{1}{\beta}\sum_{-\infty}^{\infty}D(\mathbf{q}\lambda,\omega_m)e^{-i\omega_m\tau}, \omega_m = 2\pi m/\beta \tag{1.20}$$

$$D(\mathbf{q}\lambda,\omega_m) = \int_0^\beta D(\mathbf{q}\lambda,\tau)e^{i\omega_m\tau}d\tau \tag{1.21}$$

Here we calculate the free phonon green function first. Since $H = \sum_{\mathbf{q}\lambda} \hbar\omega_{\mathbf{q}\lambda}(a^\dagger_{\mathbf{q}\lambda}a_{\mathbf{q}\lambda} + 1/2)$, we get

$$a_{\mathbf{q}\lambda}(\tau) = e^{H\tau}a_{\mathbf{q}\lambda}(0)e^{-H\tau} = e^{-\omega_{\mathbf{q}\lambda}\tau}a_{\mathbf{q}\lambda}, \tag{1.22}$$

$$a^\dagger_{-\mathbf{q}\lambda}(\tau) = e^{H\tau}a^\dagger_{-\mathbf{q}\lambda}e^{-H\tau} = e^{\omega_{\mathbf{q}\lambda}\tau}a_{-\mathbf{q}\lambda} \tag{1.23}$$

where we used $e^A C e^{-A} = C + [A, C] + \frac{1}{2!}[A, [A, C]] + \frac{1}{3!}[A, [A, [A, C]]] + ...$, or equivalently, $\dot{a}_{\mathbf{q}\lambda}(\tau) = [H, a_{\mathbf{q}\lambda}(\tau)], \dot{a}^\dagger_{-\mathbf{q}\lambda}(\tau) = [H, a^\dagger_{-\mathbf{q}\lambda}(\tau)]$.

Plug in the definition for Green's function, we reach

$$D^{(0)}(\mathbf{q}, \tau) = -\langle T_\tau \phi_{\mathbf{q},\lambda}(\tau)\phi^\dagger_{\mathbf{q},\lambda}(0)\rangle = -\langle T_\tau (a_q e^{-\omega_q\tau} + a^\dagger_{-q}e^{\omega_q\tau})(a_{-q} + a^\dagger_q)\rangle \tag{1.24}$$

$$= -\Theta(\tau)[(n_B(\omega_q) + 1)e^{-\omega_q\tau} + n_B(\omega)e^{\omega_q\tau}] - \Theta(-\tau)[n_B(\omega_q)e^{-\omega_q\tau} + (n_B(\omega) + 1)e^{\omega_q\tau}] \tag{1.25}$$

where we used $\langle a_q a^\dagger_q\rangle = n_B(\omega_q) + 1, \langle a^\dagger_q a_q\rangle = n_B(\omega_q)$, and please note $a_q(\tau)^\dagger \neq [a_q(\tau)]^\dagger$.

The Green function of frequency is,

$$\begin{aligned} D^0(\mathbf{q}, i\omega_m) &= \int_0^\beta d\tau D(\mathbf{q}\lambda, \tau)e^{i\omega_m\tau} \\ &= -(n_B(\omega_q) + 1)\frac{e^{\beta(i\omega_m - \omega_q)} - 1}{i\omega_m - \omega_q} - n_B(\omega_q)\frac{e^{\beta(i\omega_m + \omega_q)} - 1}{i\omega_m + \omega_q} \\ &= -(n_B(\omega_q) + 1)\frac{e^{-\beta\omega_q} - 1}{i\omega_m - \omega_q} - n_B(\omega_q)\frac{e^{\beta\omega_q} - 1}{i\omega_m + \omega_q} \\ &= \frac{1}{i\omega_m - \omega_q} - \frac{1}{i\omega_m + \omega_q} = \frac{2\omega_q}{(i\omega_m)^2 - \omega_q^2} \end{aligned} \tag{1.26}$$

### 1.1.4 Feynman rules

Treating the electron-phonon interaction as a perturbation, we can expand the electron Green's function to various orders in the pelturbation. Since the thermal average of the product of an odd number of phonon field operators is zero, only even orders in the perturbation expansion will survive. Here, we simply write the rules for calculating the electron Green's function.

- At order $2n$ in the electron-phonon interaction (since only even orders survive), draw all topologically distinct diagrams with n phonon lines, two extemal electron lines, and $2n-1$

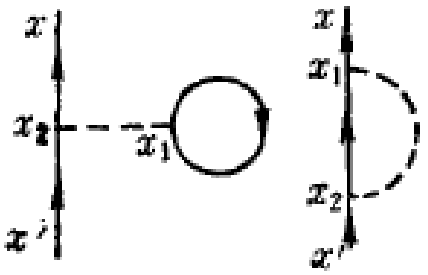



Figure 1.3: Electron self-energy due to electron-phonon interaction.

intemal electron lines.

- To each electron line of coordinates assigning $G^0(k\sigma, \omega_n)$ .
- To each phonon line of coordinates assigning $g_q^2 D^0(q, \omega_m)$
- At each vertex, conserve wave vector, frequency, and spin.
- Sum over all internal coordinates.
- Multiply each electron loop by -1
- Multiply by the factor $-1/\beta$ .



The typical Feynman diagrams are shown in Fig. . All Feymann diagrams are similar to the discussion for two-body interaction, by replacing $V(x - x') \to M_q^2 D(x - x')$. The discussion on Dyson equation is quite similar as before.

## 1.2 Electron self-energy

We calculate the zero-temperature case first, and then the finite temperature case.

We consider the lowest-order self-energy (There are two Feynman diagrams, and the Hartee-

like term with $q = 0$ vanishes due to interaction. )

$$\begin{aligned}
\Sigma(Ep) &= \frac{ig^2}{(2\pi)^4}\int d\omega d^3k \frac{1}{E-\omega-\xi(p-k)+i\delta sgn(\xi(p-k))}\frac{c^2k^2}{\omega^2-c^2k^2+i\delta} \\
&= \frac{-g^2}{(2\pi)^3}[\int_{\xi<0}\frac{d^3k}{E+ck-\xi(p-k)-i\delta}\frac{c^2k^2}{(-2)ck} - \int_{\xi>0}\frac{d^3k}{E-ck-\xi(p-k)+i\delta}\frac{c^2k^2}{2ck}] \\
&= \frac{cg^2}{16\pi^3}[\int_{|p-k|>p_F}\frac{kd^3k}{E-ck-v_F(|p-k|-p_F)+i\delta} + \int_{|p-k|<p_F}\frac{kd^3k}{E+ck-v_F(|p-k|-p_F)-i\delta}] \\
&= \frac{-cg^2}{8\pi^2 p}[\int_{p_1>p_F}\frac{k^2dkdp_1p_1}{E-ck-v_F(p_1-p_F)+i\delta} + \int_{p_1<p_F}\frac{k^2dkdp_1p_1}{E+ck-v_F(p_1-p_F)-i\delta}] \\
&= \frac{-cg^2}{8\pi^2}[\int_{p_1>p_F}\frac{k^2dkdp_1}{E-ck-v_F(p_1-p_F)+i\delta} + \int_{p_1<p_F}\frac{k^2dkdp_1}{E+ck-v_F(p_1-p_F)-i\delta}] \quad (1.27)
\end{aligned}$$

where we used $p_1^2 = |p-k|^2 = p^2+k^2-2pkx$ and $d^3k = 2\pi k^2 dkdx$ and $p_1dp_1 = -pkdx$. At the last step, we further assume $p_1 \approx p \approx p_F$. In the first line, we used the contour integral, where we have poles at $\omega_1 = E\xi(p-k)+i\delta sign\xi(p-k)$ and $\omega_{2,3} = \pm(ck-i\delta)$, and we can close integration contour in such a way, that only one of the poles of phonon Green's function is inside.

The imaginary part of self-energy is

$$Im\Sigma(Ep) = \frac{cg^2}{8\pi}[\int_{p_1>p_F}\delta(E-ck-v_F(p_1-p_F))k^2dkdp_1 - \int_{p_1<p_F}\delta(E+ck-v_F(p_1-p_F))k^2dkdp_1] \quad (1.28)$$

Next we discuss it in two limiting cases $E \ll \omega_D$ and $E \gg \omega_D$ ($\omega_D$ is the Debye temperature):

For $E \ll \omega_D$, for a given $E > 0$, only first term contributes, and we perform the integration over in the region of $k < E/c$:

$$Im\Sigma(E) = \frac{cg^2}{8\pi}\int_{ck<E}\frac{1}{v_F}k^2dk = \frac{g^2cE^3}{24\pi v_F c^3} \quad (1.29)$$

In the case of $E < 0$ we get the same result due to particle-hole symmetry. Now we see that for $E \to 0$, $Im\Sigma(E) \ll E$, so that electron-phonon interaction doesnot destory Fermi-liquid behavior, because $\sim E^3$ is negligibly small compared with electron-electron scattering contribution $\sim E^2$.

For $E \gg \omega_D$, the integration over $p_1$ doesnot put any limitation on k integration, and we

obtain $(0 < k < k_D)$

$$Im\Sigma(E) = \frac{g^2 k_D^3}{c} 24\pi v_F. \tag{1.30}$$

Next we can also calculate the real part $Re\Sigma(E)$. Starting from Eq. 1.27:

$$\begin{aligned} Re\Sigma(Ep) &= \frac{-cg^2}{8\pi^2}[\int_{p_1>p_F} \frac{k^2 dk dp_1}{E - ck - v_F(p_1 - p_F)} + \int_{p_1<p_F} \frac{k^2 dk dp_1}{E + ck - v_F(p_1 - p_F)}] \\ &= \frac{-cg^2}{8\pi^2} \int_{k<k_D} dk k^2 I_1(k) \\ I_1(k) &= \int_{p_1>p_F} \frac{dp_1}{E - ck - v_F(p_1 - p_F)} + \int_{p_1<p_F} \frac{dp_1}{E + ck - v_F(p_1 - p_F)}] \\ &= \frac{1}{v_F} \ln|\frac{E + ck}{E + ck + v_F p_F}| + \frac{1}{v_F} \ln|\frac{E - ck - v_F(p^* - p_F)}{E - ck}| \end{aligned} \tag{1.31}$$

Formally, the first integral here diverges, but this divergence is unphysical, as for large differences between p and pp we have to take into account the deviations from the linearized form electron spectrum we are using (and also the finiteness of the bandwidth). Thus we may just cut-off integration at $p_1 = p^* - p_F$. Exact value of this cut-off parameter is unimportant, as does not influence the form of the spectrum, but only renormalizes the chemical potential (contributing only to $Re\Sigma(0)$):

$$Re(\Sigma(E) - \Sigma(0)) = \frac{cg^2}{8\pi^2} \int_{k<k_D} dk k^2 \frac{m}{p_F} \ln|\frac{E - ck}{E + ck}| \tag{1.32}$$

Characteristic property of an electron self-energy due to electron-phonon interaction is its independence of momentum p. This is due to the "slowness" of phonons, compared to electrons, which leads to the local nature of the processes of phonon emission and absorption by electrons. Let us again analyze limiting cases of $E \ll \omega_D$ and $E \gg \omega_D$.

In the limit of $E \ll ck$, $(\ln|\frac{1-x}{1+x}| \approx -2x\ )$

$$Re(\Sigma(E) - \Sigma(0)) = -\frac{2mg^2 E}{8\pi^2 p_F} \int_0^{k<k_D} dk k = -\frac{mg^2 k_D^2 E}{8\pi^2 p_F} \tag{1.33}$$

In the limit of $E \gg ck$,

$$Re(\Sigma(E) - \Sigma(0)) = -\frac{2mc^2 g^2}{4\pi^2 p_F} \int_0^{k<k_D} dk k^3/E = -\frac{mg^2 c^2 k_D^4}{16\pi^2 p_F E} \tag{1.34}$$

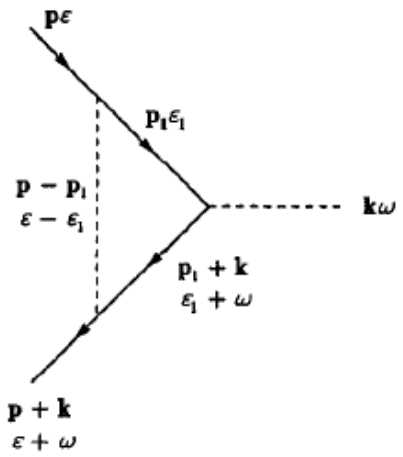



Figure 1.4: Lowest-order vertex due to electron-phonon interaction.

Quasiparticle spectrum for the region of $E \ll \omega_D$ is determined from the equation ($E = \frac{p^2}{2m^*}, \xi(p) = \frac{p^2}{2m}$):

$$E - \xi(p) = Re(\Sigma(E) - \Sigma(0)) \Rightarrow \frac{m^*}{m} = 1 + \frac{mg^2k_D^2}{8\pi^2 p_F} = 1 + \lambda \tag{1.35}$$

where $\lambda$ is called mass renormalization factor. We see that due to electron-phonon interaction an electron becomes "heavier".

The calculation of the electron self-energy can be also performed at finite temperature. The readers can check it by interests.

## 1.3 Migdal theorem

Up to now we limited ourselves to the simplest contribution to electron self energy. It may seem that we have to add also numerous diagrams with higher-order vertex corrections. But in fact we do not need these, as in the case of electron-phonon interaction all these corrections are small over the adiabaticity parameter $\omega_D/E_F \sim \sqrt{m/M} \ll 1$. This statement is usually referred to as Migdal theorem (A.B.Migdal, 1957).

Let us write down an analytic expression, corresponding to this diagram in Fig. 1.4:

$$\Gamma^1 = -g^3 \int G(p_1, E_1)G(p_1 + k, E_1 + \omega)D(E - E_1, p - p_1)\frac{d^3p_1 dE1}{(2\pi)^4} \tag{1.36}$$

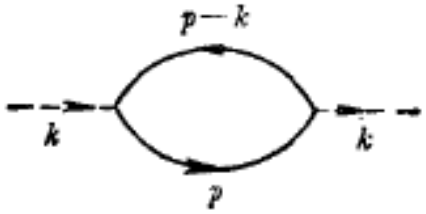



Figure 1.5: Phonon polarization function due to electron-phonon interaction.

Now make a crude estimate of this expression. The phonon propagator decreases quadratically for $|E - E_1| \gg \omega_D$, we see that the main contribution to the integral comes from the region $|E - E_1| \sim \omega_D$:

$$\Gamma^1 \sim -g^3\omega_D \int \frac{d^3p_1}{(E_1 - \xi(p_1) + i\delta sgn\xi(p_1))(E_1 + \omega - \xi(p_1 + k) + i\delta sgn\xi(p_1 + k))} \tag{1.37}$$

Consider now the remaining integral over $p_1$. Characteristic momentum transfer here is also of the order $k_D \sim p_F$. Thus we may estimate all denominators to be of the order of $\sim E_F$, and $\int d^3p_1 \sim p_F^3$. Then we have

$$\frac{\Gamma^1}{\Gamma^0} \sim g^2\omega_D \frac{p_F^2 E_F}{v_F E_F^2} \sim g^2 p_F^2 \frac{\omega_D}{v_F E_F} \sim \frac{\omega_D}{E_F} \sim \sqrt{\frac{m}{M}} \ll 1 \tag{1.38}$$

(For normal metal, $\omega_D \sim 100 - 1000K$, but $E_F \sim 10eV$.) Electrons are much lighter than ions, so this correction is practically negligible. Migdal theorem is very important, as it allows us to neglect numerous diagrams, without assumption of smallness of electron-phonon coupling.

## 1.4 Self-energy and spectrum of phonons

Return now to the analysis of Dyson equations for the phonon Green's function, which determine the phonon spectrum renormalization due to electron-phonon interaction in metals. Using the simplest approximation for the polarization operator of electron gas, we can write:

$$\begin{aligned} g^2\Pi_0(\omega, \mathbf{k}) &= -\frac{2ig^2}{(2\pi)^4} \int \frac{dEd^3p}{[E - \xi(p) + i\delta sgn(\xi(p))][E + \omega - \xi(p + k) + i\delta sgn(\xi(p + k))]} \\ &= -\frac{g^2 m p_F}{\pi^2}\{1 - \frac{\omega}{2v_F k}\ln|\frac{\omega + v_F k}{\omega - v_F k}| + \frac{i\pi|\omega|}{2v_F k}\theta(1 - \frac{|\omega|}{vk})\} \end{aligned} \tag{1.39}$$

Additionally, the phonon Green's function in the system with electron-phonon interaction is

determined by Dyson equation of the form:

$$D^{-1}(\omega,\mathbf{k}) = D_0^{-1}(\omega,\mathbf{k}) - g^2\Pi(\omega,\mathbf{k})$$
$$= \frac{\omega^2 - c_0^2k^2}{c_0^2k^2} + 2\zeta \tag{1.40}$$

where we have used the the static approximation: $\omega = 0$: $g^2\Pi_0 \approx -\frac{g^2mp_F}{\pi^2} = -2\zeta$. $c_0$ is "bare" sound velocity, while the renormalized phonon spectrum is written as $\omega = ck$, which is from $D^{-1}(\omega,k) = 0$:

$$c^2 = c_0^2(1-2\zeta) \tag{1.41}$$

We see the electron-phonon interaction leads to the "softening" of the lattice (decrease of phonon frequency). Furthermore, if the frequency becomes negative, it indicates the lattice is unstable.

To find phonon damping we have to take into account the imaginary part of polarization operator

$$g^2Im\Pi_0(\omega,\mathbf{k}) = -\pi\zeta\frac{|\omega|}{v_Fk} \tag{1.42}$$

Inserting this into Dyson equation for the phonon Green's function and seeking the solution for the spectrum as $\omega = ck + i\gamma$, we find:

$$\gamma = \frac{\pi}{2}\zeta\frac{c^2k}{v_F} = \frac{\pi}{2}\zeta\frac{c\omega}{v_F} \tag{1.43}$$

In the previous chapter we have noted, that at $q = 2p_F$ polarization operator $\Pi_0(q,0)$ has the logarithmic singularity. This singularity becomes stronger in two-dimension and one-dimension. To see it,

$$D(\omega,q) = \frac{1}{D^{-1}(\omega,q) - g^2\Pi_0(\omega,q)} = \frac{\omega_0^2(q)}{\omega^2 - \omega_0^2(q) - g^2\omega_0^2(q)\Pi_0(\omega,q)} \tag{1.44}$$

so that the phonon spectrum is $\omega^2 = \omega_0^2(q)(1 + g^2\Pi_0(\omega,q))$. Due to $\Pi_0(q,0) \to -\infty$ at $q = 2p_F$ for one-D, the frequency $\omega^2 < 0$ (so frequency becomes imaginary) is obtained for any value of coupling constant $g$. This signifies an instability of the system, leading to the apprearance of spontaneous static deformation of the lattice with the wave vector $Q = 2p_F$. This phenomenon is

called Peierls instability. In general case, a special property of the Fermi surface is needed for the appearance of such "giant" anomalies, which is called "nesting". Nesting property of the Fermi surface means that certain parts of the Fermi surface are congruent (completely coincide with each other) after the translation by some specific vector Q in momentum space (vector of nesting): $E(p+Q)-\mu = -E(p)+\mu$. In this case, $\Pi_0(q,0)$ prosseses a singularity $\Pi_0(q,0) \sim \ln|q-Q|$. In all cases with nesting, calculation of polarization operator shows thedivergence at $q = Q$, leading to the appearance of the giant Kohn anomaly, in phonon spectrum and lattice instability (structural phase transition, leading to static superstructure with wave vector Q.)

## 1.5 Plasma model

Let us consider now the simplest "plasma" model of a metal, where both phonons and electron phonon interactions appear self-consistently. Start with plasma consisting of electrons and ions, interacting via (non-screened) Coulomb forces. In first approximation, collective oscillations in this system are just independent plasma oscillations of electrons and ions. We shall show how the account of screening allows to introduce the "usual" phonons and obtain the coherent description of electron-phonon interaction.

The Hamiltonianof electron-ion plasma as

$$H = \sum_k E_k c_k^+ c_k + \sum_{q,\lambda} \Omega_{q\lambda}(a_{q\lambda}^+ a_{q\lambda} + \frac{1}{2}) + \sum_{k,k,',\lambda} g_{k,k'} c_k^+ c_{k'}(a_{k-k',\lambda} + a_{k'-k,\lambda}^+) + \frac{1}{2}\sum_{p,k,q} V_q c_{p+q}^+ c_{k-q}^+ c_k c_p \tag{1.45}$$

where $E_k$ is the energy of electron, and $V_q = 4\pi e^2/q^2$. In the jellium model we assume ions to form a homogeneous structureless medium, so that $\Omega_{q,\lambda}^2 = 4\pi n(Ze)^2/M$, where $n$ is ion density, Z ion charge, M ion mass. In jellium model this is the only mode of ion oscillations. The bare electron-phonon coupling is defined as

$$g_{k,k',\lambda} = -(\frac{n}{M\Omega_{k\lambda}^2})^{1/2} < k'|\nabla_i V_{ei}|k > e_{q\lambda}, q = k - k' \tag{1.46}$$

where $e_{q\lambda}$ is polarization vector of bare phonons (see Eq.1.1.2). And we know $g_{k,k',\lambda}^2 \sim \frac{1}{(k-k')^2}$.

Now we have to make renormalizations, accounting for screening and regularizing such sin-

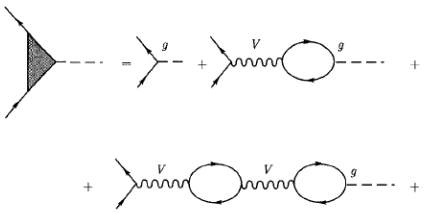


Figure 1.6: Screening of electron-phonon vertex.

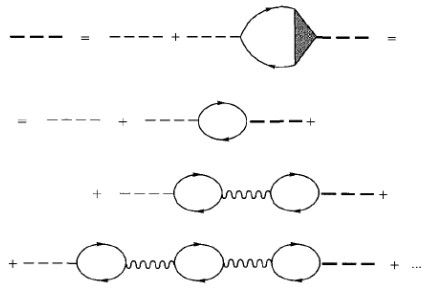


Figure 1.7: Dyson equation for phonon green's function.

gularities. For Coulomb interaction between electrons we can just use the RPA expression:

$$V(q,\omega) = \frac{4\pi e^2}{q^2\epsilon(q\omega)}, \epsilon(q\omega) = 1 - \frac{4\pi e^2}{q^2}\Pi_0(q\omega) \tag{1.47}$$

where the dielectric function of free electrons is from the bubble diagram. In a similar way, as shown in Fig. 1.6, we can describe the screening of electron-phonon vertex as

$$\tilde{g}(q,\lambda) = g + gV_q\Pi_0 + gV_q\Pi_0V_q\Pi_0 + ... = \frac{g(q,\lambda)}{\epsilon(q\omega)} \tag{1.48}$$

To define the "physical" phonon spectrum we can write Dyson equation, as in Fig. 1.7,

$$\begin{aligned} D^{-1}(q\lambda,\omega) &= D_0^{-1}(q\lambda,\omega) - g^2\Pi_0(q,\omega) - g^2\Pi_0(q\omega)V_q\Pi_0(q\omega) - ... \\ &= D_0^{-1}(q\lambda,\omega) - \frac{g^2(q,\lambda)}{V_q}(\frac{1}{\epsilon(q\omega)} - 1) \\ \Rightarrow D(q\lambda,\omega) &= \frac{\Omega_{q\lambda}^2}{\omega^2 + i\delta - \frac{g^2\Omega_{q\lambda}^2}{V_q\epsilon(q\omega)} - \Omega_{q\lambda}^2(1-\frac{g^2}{V_q})} = \frac{\Omega_{q\lambda}^2}{\omega^2 + i\delta - \omega^2(q\lambda)} \end{aligned} \tag{1.49}$$

where

$$\omega^2(q\lambda) = \Omega^2(1 - \frac{g^2}{V_q}) + \frac{g^2\Omega^2}{V_q\epsilon} \tag{1.50}$$

Moreover, in this model we can also determine the full (effective) inter-electron interaction, which is necessary for calculations of superconducting properties of metals (as discussed below). This interaction can be described by diagrams shown in Fig. 1.8 and is given by:

$$\begin{aligned} V_{eff}(q\omega) &= \frac{4\pi e^2}{q^2\epsilon(q\omega)} + \frac{g^2(q,\lambda)}{\epsilon^2(q,\omega)}\frac{\Omega_{q\lambda}^2}{\omega - \omega^2(q\lambda)} \\ &= \frac{4\pi e^2}{q^2\epsilon_{eff}(q\omega)}, \end{aligned} \tag{1.51}$$

where $\omega^2(q\lambda)$ is the spectrum of renormalized phonons. Here the dielectric function contains both electron-electron and electron-phonon contributions.

Next we can make some analysis based on the dielectric function,

$$\frac{1}{\epsilon_{eff}(q\omega)} = \frac{1}{\epsilon(q\omega)} + \frac{g^2(q,\lambda)}{V_q\epsilon(q\omega)^2}\frac{\Omega_{q\lambda}^2}{\omega^2 - \omega^2(q\lambda)} \tag{1.52}$$

Stability of the lattice requires $\omega^2(q\lambda) > 0$ (see Eq. 1.50) and $\epsilon_{eff} < 0$ give the condition of

$$1 - \frac{g^2(q,\lambda)}{V_q}(1 - \frac{1}{\epsilon(q,0)}) > 0 \tag{1.53}$$

$$\frac{1}{\epsilon_{eff}(q\omega)} < 0. \tag{1.54}$$

These expressions allow to determine conditions, when this effective interaction may become attractive, which is a necessary condition for the appearance of superconductivity. Or, one see that, taking into account electron-phonon interaction, we see effective electron-electron interaction could become attractive.

Physically, this attractive interaction comes from the phonon progragator. This can be even seen from the bare phonon Green's function: $g^2 D^0 \sim g^2 \frac{\omega_q^2}{\omega^2 - \omega_q^2}$. If two electrons close to the Fermi surface, the energy transfer $\omega \sim E_1 - E_2 = 0$, so the effective electron-electron interaction becomes negative.

The physical reason is, the lattice has a large atomic mass and a certain inertia, so the effective electron-electron interaction due to electron-lattice scattering has a delayed effect and is no longer a transient interaction.

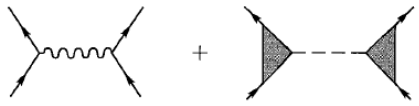


Figure 1.8: Electron-electron vertex due to phonon scattering.

## 1.6 Cooper-pair instability

Motivated by the discussion above, we consider the following simplified model for superconductivity:

$$H_{el-ph} = -V \int d\mathbf{r} \psi^+_\sigma(\mathbf{r}) \psi^+_{\sigma'}(\mathbf{r}) \psi_{\sigma'}(\mathbf{r}) \psi_\sigma(\mathbf{r}) \tag{1.55}$$

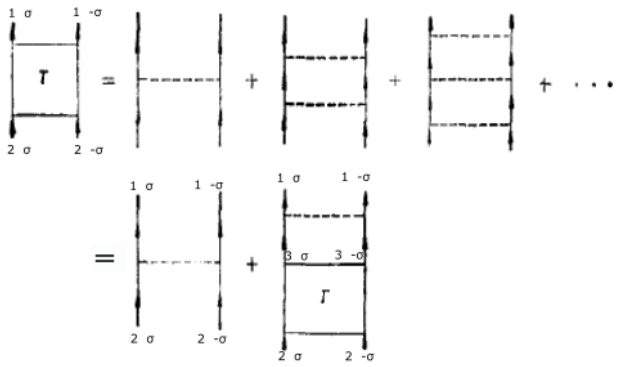
where the attractive interaction comes from electron-phonon interactions. The attractive interaction is crucial for the formation of the Cooper pair. This leads to the general idea that electrons in metals with opposite momenta and spins (Pauli principle!) attract each other due to phonon exchange, which is the basic concept of BCS theory to superconductivity [J. R. Schrieffer (1964)].

To discuss the Cooper-pairs (bound state formed by two electrons), one could use the Bogovliubov mean-field theory. This is the simplest way. One could consult many other textbooks. Here, we would like to explore another way, the cooper pair instability, or "ladder" diagrams, describing interaction of two quasiparticles (electrons) close to the Fermi surface. In specific, we discuss two-particle Green's function or vertex, as defined as

$$\Gamma(1,2) = (-i)^2 \langle T \psi_\uparrow(r_1,t_1) \psi_\downarrow(r_1,t_1) \psi^+_\downarrow(r_2,t_2) \psi^+_\uparrow(r_2,t_2) \rangle. \tag{1.56}$$

and we mainly concern the Fourier transformation $\Gamma(\mathbf{r},t) = \sum_n \int \frac{d^3k}{(2\pi)^3} \Gamma(\mathbf{p},\omega_n) \exp(i(\mathbf{p}\cdot\mathbf{r} - \omega_n t))$.

In the propragation, electrons must encounter multi-times scattering, parts of which can be expressed by the following ladder diagrams: which relates to a self-consistent equation:




$$\Gamma(1,2) = \Gamma_0(1,2) + \int d(3)\Gamma(1,3)\Gamma_0(3,2) \tag{1.57}$$

$$\rightarrow \Gamma(\mathbf{p}, i\omega_n) = -V - iV\sum_l \int d^3\mathbf{q} G(\mathbf{p}-\mathbf{q}, i\omega_n - i\omega_l)G(\mathbf{q}, i\omega_l)\Gamma(\mathbf{p}, i\omega_n) \tag{1.58}$$

$$\rightarrow \Gamma(\mathbf{p}, i\omega_n)[1 + iV\sum_l \int d^3\mathbf{q} G(\mathbf{p}-\mathbf{q}, i\omega_n - i\omega_l)G(\mathbf{q}, i\omega_l)] = -V \tag{1.59}$$

$$\rightarrow \Gamma(\mathbf{p}, i\omega_n) = -V[1 + iV\sum_l \int d^3\mathbf{q} G(\mathbf{p}-\mathbf{q}, i\omega_n - i\omega_l)G(\mathbf{q}, i\omega_l)]^{-1} \tag{1.60}$$

The pair instability relates to the singularity of $\Gamma$, which also determines the binding energy and critical temperature. The singularity is given by the zero of denominator:

$$0 = 1 + iV\sum_l \int d^3\mathbf{q} G(\mathbf{p}-\mathbf{q}, i\omega_n - i\omega_l)G(\mathbf{q}, i\omega_l) \tag{1.61}$$

This singularity leads to the instability of Fermi-liquid, and drives to superconductivity. Next we calculate the integral out step by step. The Mastubara summation gives

$$\begin{aligned} &-i\sum_l \int d^3\mathbf{q} G(\mathbf{p}-\mathbf{q}, i\omega_n - i\omega_l)G(\mathbf{q}, i\omega_l) \\ &= -i\sum_l \int d^3\mathbf{q} \frac{1}{i\omega_n - i\omega_l - E_{\mathbf{p}-\mathbf{q}}} \frac{1}{i\omega_l - E_{\mathbf{q}}} = \int d^3\mathbf{q} \frac{n_F(E_{\mathbf{p}-\mathbf{q}}) - n_F(E_{\mathbf{q}})}{i\omega_n - E_{\mathbf{p}-\mathbf{q}} - E_{\mathbf{q}}} \end{aligned} \tag{1.62}$$

Here we hope to get some "static" estimation, which can help us simplify the discussion greatly. We can set $\mathbf{p} \rightarrow 0$ first. Next we use the relation $f(-E) = 1 - f(E), 1 - 2f(E) = \tanh(\beta E/2)$, and the equation Eq. 1.61 becomes

$$\frac{1}{V} = -\int' d^3\mathbf{q} \frac{1 - 2f(E_{\mathbf{q}})}{i\omega_n - 2E_{\mathbf{q}}} \tag{1.63}$$

where we limit the integral around the Fermi surface. Please note the negative sign here, which

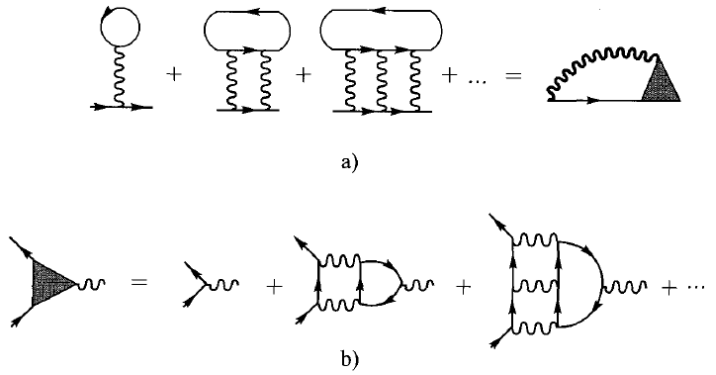



Figure 1.9: Electron-electron vertex due to phonon scattering.



is important. Next, after analytically continuation $i\omega_n \to z + i0^+$, and then set $z \to 0$, we have

$$\frac{1}{V} = N(E_F) \int_{-\omega_D}^{\omega_D} dE \frac{1 - 2f(E)}{2E} \tag{1.64}$$

To deal with this integral, we check the two separated integral regime:

$$\begin{cases} \int_{-T}^{T} dE \frac{1-2\frac{1}{e^{\beta E}+1}}{2E} \approx \int_{-T}^{T} dE \frac{1-2\frac{1}{\beta E+2}}{2E} \sim \frac{\beta 2T}{4} \sim const, |E| < T \\ \int_{T}^{\omega_D} dE \frac{1-2\frac{1}{e^{\beta E}+1}}{2E} \approx \int_{T}^{\omega_D} dE \frac{1}{2E} = \ln \frac{\omega_D}{T}, |E| > T \end{cases} \tag{1.65}$$

So we approximately have

$$\frac{1}{N(E_F)V} \approx \ln \frac{\omega_D}{k_B T_c} \Rightarrow k_B T_c = \omega_D e^{-\frac{1}{VN(E_F)}} \tag{1.66}$$

$\omega_D$ is Debye frequency, playing the role of cut-off energy here.

Finally, the vertex shows Cooper instability the pole in the vertex part in the upper half-plane of frequency formally signifies the appearance of an unstable collective mode with exponentially growing (in time) amplitude. The temperature, corresponding to the appearance of this instability, defines the temperature of superconducting transition.

In addition, Cooper "ladder" contributes to electron self-energy via diagrams shown in Fig. 1.9.

## 1.7 Ginzburg-Laudau expansion

Here, we turn to the phenomenological approach to superconductivity [ Ginzburg and Landau (1950)], and based on the expansion of free energy in powers of the order parameter, allowing to describe main properties of superconductors close to superconducting transition temperature [ De Gennes 1966]. It is surprising that the rich phenomenology of the superconducting state could be quantitatively described by the GL theory, even without knowledge of the underlying microscopic mechanism based on the BCS theory. It is based on the idea that the superconducting transition is one of the second order phase transition. The order parameter is described by two components (complex number $\psi = |\psi|e^{i\theta}$ ). The amplitude $|\psi|$ is zero in the normal phase above a superconducting transition temperature $T_c$ and is finite in the superconducting phase below $T_c$. in some limiting cases, the order parameter $|\psi|^2$ of the GL theory is proportional to the pair potential $|\Delta|^2$. The parameter $|\psi|^2$ represents the local density of superconducting electrons, $n_s$. At the same time this also shows that $q^* = 2e(< 0)$ and $m^* = 2m$.

We start with the Hamiltonian with attractive interaction again:

$$H = \int d\mathbf{r}\psi_\sigma^+(\mathbf{r})[-\frac{1}{2m}\nabla^2 - \mu]\psi_\sigma(\mathbf{r}) - V\int d\mathbf{r}\psi_\sigma^+(\mathbf{r})\psi_{\sigma'}^+(\mathbf{r})\psi_{\sigma'}(\mathbf{r})\psi_\sigma(\mathbf{r}) \tag{1.67}$$

As has already been mentioned, this is the effective Hamiltonian describing the low-energy states in a width of order $\omega_D$ in the vicinity of the Fermi surface.

The quantum partition function is given by

$$Z = \int D\overline{\psi}_\sigma D\psi_\sigma e^{-S[\overline{\psi}_\sigma,\psi_\sigma]} = \int D\overline{\psi}_\sigma D\psi_\sigma D\Delta D\overline{\Delta} e^{-S_{eff}[\overline{\psi}_\sigma,\psi_\sigma,\Delta,\overline{\Delta}]} \tag{1.68}$$

$$S[\overline{\psi}_\sigma,\psi_\sigma] = \int_0^\beta d\tau\int d\mathbf{r}\left[\overline{\psi}_\sigma\partial_\tau\psi_\sigma + H[\overline{\psi}_\sigma,\psi_\sigma] - \mu N[\overline{\psi}_\sigma,\psi_\sigma]\right] \tag{1.69}$$

$$\begin{aligned} S_{eff}[\overline{\psi}_\sigma,\psi_\sigma,\Delta,\overline{\Delta}] = &\int_0^\beta d\tau\int d\mathbf{r}\overline{\psi}_\sigma[\partial_\tau - \frac{1}{2m}\nabla^2 - \mu]\psi_\sigma \\ &+ \overline{\Delta}(\mathbf{r},\tau)\psi_\downarrow(\mathbf{r},\tau)\psi_\uparrow(\mathbf{r},\tau) + \Delta(\mathbf{r},\tau)\overline{\psi}_\downarrow(\mathbf{r},\tau)\overline{\psi}_\uparrow(\mathbf{r},\tau) + \frac{1}{V}\overline{\Delta}(\mathbf{r},\tau)\Delta(\mathbf{r},\tau) \end{aligned} \tag{1.70}$$

where we have used the Hubbard Stratonovich transformation. $\psi, \overline{\psi}$ can be taken as the Grassmann fields related to the fermionic fields.

Next we transfer to the momentum space and set $\Delta(\mathbf{r},\tau) = \Delta$. We will integrate out the fermion field (see Gaussian integral formula below) we can write the action using the Green's

function

$$S_{eff}[\overline{\psi}_\sigma, \psi_\sigma, \Delta, \overline{\Delta}] = \frac{1}{V}\overline{\Delta}\Delta + \sum_{\omega_n}\sum_{\mathbf{k}}(\overline{\psi}_\uparrow(k), \psi_\downarrow(-k)) \begin{pmatrix} -i\omega_n + \xi_k & \Delta \\ \overline{\Delta} & -i\omega_n - \xi_k \end{pmatrix} \begin{pmatrix} \psi_\uparrow(k) \\ \overline{\psi}_\downarrow(-k) \end{pmatrix} \tag{1.71}$$

$$= \frac{1}{V}\overline{\Delta}\Delta - \sum_{\omega_n}\sum_{\mathbf{k}} \ln\det \begin{pmatrix} -i\omega_n + \xi_k & \Delta \\ \overline{\Delta} & -i\omega_n - \xi_k \end{pmatrix} \tag{1.72}$$

$$= \frac{1}{V}\overline{\Delta}\Delta - \sum_{\omega_n}\sum_{\mathbf{k}} \ln(-\omega_n^2 - \xi_k^2 - \overline{\Delta}\Delta) \tag{1.73}$$

$$= \frac{1}{V}\overline{\Delta}\Delta - \sum_{\omega_n}\sum_{\mathbf{k}} \ln(-\omega_n^2 - \xi_k^2) - \sum_{\omega_n}\sum_{\mathbf{k}} \ln(1 + \frac{\overline{\Delta}\Delta}{\omega_n^2 + \xi_k^2}) \tag{1.74}$$

Grassmann Gaussian integration.— The Grassmann Gaussian integration formula:

$$\int d\overline{\psi}d\psi e^{-\overline{\psi}a\psi} = a. \tag{1.75}$$

For a multi-dimensional vector of Grassmann variables,

$$\int d\overline{\psi}d\psi e^{-\overline{\psi}\cdot\mathbf{A}\cdot\psi} = \det \mathbf{A}. \tag{1.76}$$



If assuming $\Delta$ is a small number, we can expand it in series ($\ln x = x - x^2/2...$). The fourth order term is

$$\frac{1}{2}\sum_{\omega_n}\sum_{\mathbf{k}} \frac{(\overline{\Delta}\Delta)^2}{(\omega_n^2 + \xi_k^2)^2} \equiv b(T)(\overline{\Delta}\Delta)^2, b(T) > 0. \tag{1.77}$$

The second order term is

$$a(T)\overline{\Delta}\Delta = (\frac{1}{V} - \sum_{\omega_n}\sum_{\mathbf{k}} \frac{1}{\omega_n^2 + \xi_k^2})\overline{\Delta}\Delta \tag{1.78}$$

The second term has a minus sign, due to the attractiveness of the force. Doing the Matsubara

summation,

$$a(T) = \frac{1}{V} - \sum_{\mathbf{k}} \frac{f(-\xi_k) - f(\xi_k)}{2\xi_k} = \frac{1}{V} - N(E_F) \int_{\omega_D}^{-\omega_D} d\xi \frac{1 - 2f(\xi)}{2\xi} \tag{1.79}$$

from which we determine the critical temperature at $a(T_c) = 0$:

$$k_B T_c = \omega_D e^{-\frac{1}{VN(E_F)}} \tag{1.80}$$

which should be the same as Eq. 1.66. Near the critical temperature, we expand it

$$\begin{aligned} a(T) = a(T) - a(T_c) &= N(E_F) \int_{\omega_D}^{-\omega_D} d\xi \frac{f(\xi, T) - f(\xi, T_c)}{\xi} \\ &= (T - T_c) N(E_F) \int_{\omega_D}^{-\omega_D} \frac{d\xi}{\xi} \frac{\partial f}{\partial T}|_{T \to T_c} \approx (T - T_c) N(E_F) \int_{\omega_D}^{-\omega_D} \frac{d\xi}{\xi} \frac{\partial f}{\partial \xi}|_{T \to T_c} \frac{\xi}{T_c} \\ &\sim N(E_F) \frac{T - T_c}{T_c} \end{aligned} \tag{1.81}$$

To sum up, we reach the so-called Ginzburg- Landau theory of superconductivity:

$$S_{eff} = a(T) \overline{\Delta} \Delta + b(T) (\overline{\Delta} \Delta)^2 + |\partial \Delta|^2 \tag{1.82}$$

For $a(T) > 0$ and $T > T_c$, the potential has a stable minimum at $\Delta = 0$. On the other hand, for $a(T) < 0$ and $T < T_c$, a minimum at a finite $\Delta$ arises. The spatial dependence of order parameter describes the fluctuations. One could derive it from the scratch (we omit it here). Thus, one can understand the transition to superconductivity within the Landau theory of phase transition.

The superconductivity Cooper pair amplitude $\Delta_0$ is determined by the condition:

$$\frac{\partial S[\Delta, \overline{\Delta}]}{\partial \overline{\Delta}} = 0 \Rightarrow \overline{\Delta}_0 (a(T) + 2b(T) |\Delta_0|^2) = 0 \Rightarrow |\Delta_0| = \sqrt{\frac{-a(T)}{2b(T)}} \sim \sqrt{T_c - T} \tag{1.83}$$

which describes the critical behavior close to the $T_c$.

Up to now, the simple discussion on $\Delta_0$ ignores the fact that the symmetry broken by the ground state is a global $U(1)$ symmetry. We will discuss it next.

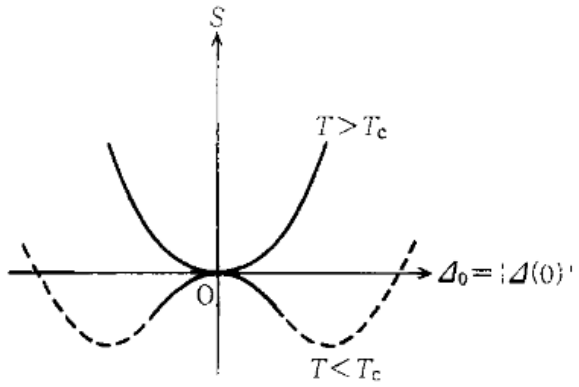



Figure 1.10: Ginzburg-Landau potential of the order parameter of the superconductivity.

### 1.7.1 Goldstone mode

The above saddle point solution only depends on the amplitude $|\Delta|^2$. One will immediately see that the gauge symmetry $\Delta \to \Delta e^{2i\theta}$ doesnot change the saddle point solution, which implies a Goldstone mode in the low-energy. Next we try to derive the low-energy theory of this Goldstone mode:

$$S[\theta] = \int d\tau \int d^d r [c_1(\partial_\tau \theta)^2 + c_2(\nabla\theta)^2] \tag{1.84}$$

The first term describes the dynamical fluctuation, and the second is the energy cost associated with phase fluctuation. If it couples with the electromagnetic potential via the minimal substitution, we have

$$S[\theta, \mathbf{A}] = \int d\tau \int d^d r [c_1(\partial_\tau \theta + \phi)^2 + c_2(\nabla\theta - \mathbf{A})^2] \tag{1.85}$$

where the coefficient $c_1$ and $c_2$ can be determined microscopically (see discussion between).

Let us go back to the effective action and include the coupling between electron and electromagnetic field:

$$\begin{aligned} S_{eff}[\overline{\psi}_\sigma, \psi_\sigma, \Delta_0, \overline{\Delta}_0] = \frac{1}{V}\overline{\Delta}\Delta + \sum_{\omega_n}\sum_{\mathbf{k}} (\overline{\psi}_\uparrow(k), \psi_\downarrow(-k)) \times \\ \begin{pmatrix} -i\omega_n - i\phi + \mu - \frac{1}{2m}(-i\nabla - \mathbf{A})^2 & \Delta_0 e^{2i\theta} \\ \overline{\Delta}_0 e^{-2i\theta} & -i\omega_n + i\phi - \mu + \frac{1}{2m}(-i\nabla - \mathbf{A})^2 \end{pmatrix} \begin{pmatrix} \psi_\uparrow(k) \\ \overline{\psi}_\downarrow(-k) \end{pmatrix} \end{aligned} \tag{1.86}$$

One can check that the phase $\theta$ can be absorbed into the gauge field $\phi \to \phi' = \phi + \partial_\tau\theta, \mathbf{A} \to$

$\mathbf{A}' = \mathbf{A} - \nabla\theta$.

We reformulate the matrix part as

$$\begin{aligned}
&\begin{pmatrix} -i\omega_n - i\phi + \mu - \frac{1}{2m}(-i\nabla - \mathbf{A})^2 & \Delta e^{2i\theta} \\ \overline{\Delta} e^{-2i\theta} & -i\omega_n + i\phi - \mu + \frac{1}{2m}(-i\nabla - \mathbf{A})^2 \end{pmatrix} \\
&= \sigma_0\partial_\tau - \sigma_3(i\phi' + \frac{1}{2m}(-i\nabla - \mathbf{A}')^2 - \mu) + \sigma_1\Delta_0 \\
&= \sigma_0\partial_\tau - \sigma_3(-\frac{1}{2m}\nabla^2 - \mu) + \sigma_1\Delta_0 - i\sigma_3\phi' + \frac{i}{2m}\sigma_0[\nabla, \mathbf{A}']_+ - \sigma_3\frac{1}{2m}\mathbf{A}'^2 \\
&\equiv G_0^{-1} + X_1 + X_2
\end{aligned} \tag{1.87}$$

By integration over the fermion fields, we have

$$\begin{aligned}
S[\mathbf{A}] &= -tr\ln[G_0^{-1} + X_1 + X_2] = const. - tr\ln(1 - G_0(X_1 + X_2)) \\
&= const. + tr(G_0X_1) + tr(G_0X_2 + \frac{1}{2}G_0X_1G_0X_1) + ... \\
&= \int d\tau \int d^dr[\frac{T}{L^d}\sum_{\mathbf{p}} \frac{-\omega_n^2 + \lambda_p^2 - 2\Delta_0^2}{(\omega_n^2 + \lambda_p^2)^2}\phi'^2 + (\frac{n}{2m} - \frac{1}{dm}\frac{T}{L^d}\sum_{\mathbf{p}} \frac{p^2(-\omega_n^2 + \lambda_p^2)}{(\omega_n^2 + \lambda_p^2)^2})\mathbf{A}'^2] \\
&\equiv \int d\tau \int d^dr[c_1(\phi + \partial_\tau\theta)^2 + c_2(\mathbf{A} - \nabla\theta)^2]
\end{aligned} \tag{1.88}$$

where $\lambda_p^2 = \Delta_0^2 + \xi_p^2$.

Usually, the coefficient $c_2 = \frac{n_s}{m}$, and $n_s$ is the superfluid density. $c_1$ is related to the charge density.

### 1.7.2 Anderson-Higgs mechanism

Consider a simplified version of action, by neglecting the fluctuations of the phase $\partial_\tau\theta = 0$, and by excluding the electric fields actong on the superconductor $\phi = 0, \partial_\tau\mathbf{A} = 0$, we have

$$S[\mathbf{A}, \theta] = \frac{\beta}{2}\int d^dr[\frac{n_s}{m}(\nabla\theta - \mathbf{A})^2 + (\nabla \times \mathbf{A})^2]. \tag{1.89}$$

By integration over $\theta$, we have the effective action

$$\begin{aligned} S[\mathbf{A},\theta] &= \sum_{\mathbf{q}}[\frac{n_s}{m}(i\mathbf{q}\theta_{\mathbf{q}} - \mathbf{A}_{\mathbf{q}})\cdot(-i\mathbf{q}\theta_{-\mathbf{q}} - \mathbf{A}_{-\mathbf{q}})] + (\mathbf{q}\times\mathbf{A}_{\mathbf{q}})\cdot(\mathbf{q}\times\mathbf{A}_{-\mathbf{q}}) \\ &= \sum_{\mathbf{q}}[\frac{n_s}{m}(q^2\theta_{\mathbf{q}}\theta_{-\mathbf{q}} - 2i\theta_{\mathbf{q}}\mathbf{q}\cdot A_{-\mathbf{q}} + \mathbf{A}_{\mathbf{q}}\cdot\mathbf{A}_{-\mathbf{q}}) + (\mathbf{q}\times\mathbf{A}_{\mathbf{q}})\cdot(\mathbf{q}\times\mathbf{A}_{-\mathbf{q}})] \\ \rightarrow S[\mathbf{A}] &= \sum_{\mathbf{q}}[\frac{n_s}{m}(\mathbf{A}_{\mathbf{q}}\cdot\mathbf{A}_{-\mathbf{q}} - \frac{(\mathbf{q}\cdot\mathbf{A}_{\mathbf{q}})(\mathbf{q}\cdot\mathbf{A}_{-\mathbf{q}})}{q^2})] + (\mathbf{q}\times\mathbf{A}_{\mathbf{q}}\cdot(\mathbf{q}\times\mathbf{A}_{-\mathbf{q}}) \\ &= \sum_{\mathbf{q}}(\frac{n_s}{m} + q^2)A^{\perp}_{\mathbf{q}}\cdot A^{\perp}_{\mathbf{q}}) \end{aligned} \tag{1.90}$$

where $A_{\mathbf{q}} = A^{\perp}_{\mathbf{q}} + A^{//}_{\mathbf{q}}$, $A^{//}_{\mathbf{q}} = \frac{\mathbf{q}(\mathbf{q}\cdot A_{\mathbf{q}})}{q^2}$.

Here we see that, starting from the Goldstone mode $\theta$ and the gauge field $\mathbf{A}$, we have arrived at an action for the gauge field $S[\mathbf{A}]$. In this context, the Goldstone mode has been absorbed into the gauge degree of freedom $\mathbf{A}$. The coupling between gauge field and Goldstone mode gives rise to a finite mass term ($\frac{n_s}{m}$) to the gauge field. In the long wave limit, the action does not vanish. This "photon field" (vector potential) has consumed the Goldstone mode to become massive. This principal mechanism was proposed by Higgs in 1964, dubbed as Anderson-Higgs mechanism. Higgs mechanism is quite general, not limit to the discussion of superconductivity.

Next, we consider the equation of motion of the vector potential:

$$(\frac{n_s}{m} + q^2)\mathbf{A}_{\mathbf{q}} = 0 \rightarrow (\frac{n_s}{m} - \nabla^2)\mathbf{A}(\mathbf{r}) = 0 \tag{1.91}$$

By applying $\nabla\times$ from the very left side, we have the London equation:

$$(\frac{n_s}{m} - \nabla^2)\mathbf{B}(\mathbf{r}) = 0 \tag{1.92}$$

which leads to the solution of

$$\mathbf{B}(\mathbf{r}) = \mathbf{B}_0\exp[-|\mathbf{r}|/\lambda] \tag{1.93}$$

where $\lambda = \sqrt{m/n_s}$ is the penetration depth. Physically, this means that a bulk superconductor is diamagnetic, i.e. magnetic field cannot goes through a superconductor (for the type-I), which is known as the Meissner effect.

Similarly, the second London equation can be derived from the effective action **A**.

# Chapter 2

# Problem Sets

## 2.1 Matsubara frequency summation problem

In the calculation of Ginzberg-Landau theory of superconductivity, we need to deal with the Matsubara frequency summation:

$$\begin{aligned} a(T) &= \frac{1}{V} - \sum_{\omega_n} \sum_{\mathbf{k}} \frac{1}{\omega_n^2 + \xi_k^2} \\ &= \frac{1}{V} - \sum_{\mathbf{k}} \frac{f(-\xi_k) - f(\xi_k)}{2\xi_k} = \frac{1}{V} - N(E_F) \int_{\omega_D}^{-\omega_D} d\xi \frac{1 - 2f(\xi)}{2\xi}. \end{aligned} \tag{2.1}$$

Please work out this frequency summation by yourself.

## 2.2 London equation problem

From the effective action

$$S[\mathbf{A}] = \sum_{\mathbf{q}} (\frac{n_s}{m} + q^2) A^{\perp}_{\mathbf{q}} \cdot A^{\perp}_{\mathbf{q}}), \tag{2.2}$$

please derive the second London equation.