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Chapter 1

Entanglement entropy

Entanglement expresses non-local connotations inherent to quantum mechanics, which has prompted remarkable insights into various fields of modern physics, bridging microscopic laws in quantum matters and macroscopic structure of space-time. Compared to the traditional methods by inspecting various (local) order parameters and their responses to external perturbations, the study of many-body wave function via entanglement-based analysis developed in quantum information science is able to unveil novel properties in a large variety of collective quantum phenomena, ranging from the presence of topological order to the onset of quantum criticality.

In this chapter, we will explore the calculation of the entanglement entropy, which is one of simple but efficient way to measure the amount of entanglement encoded in a quantum system.

1.1 A. Peschel's formula

The partial trace involved in the construction of a reduced density matrix can be turned into a path integral. However, if the trace is over fermionic degrees of freedom, the resulting path integral is Grassmannian and is hard to evaluate numerically. For free fermions, Peschel [J. Stat. Mech. P06004(2004)] found a way to derive the reduced density matrix without evaluating the Grassmannian integral. We review Peschel?s method in this section.

A Gaussian density matrix ρ, where $\ln \rho$ is bilinear in the creation and annihilation operators, is completely determined by its correlation matrix G, which itself is essentially a single-particle "density matrix",

Consider a many-body density matrix?(un-normalized)

$$\rho = e^{-\beta \hat{H}} \tag{1.1}$$

where we assume H takes the form of

$$\hat{H} = A_{ij}\varphi_i^\dagger \varphi_j + \frac{1}{2}(B_{ij}\varphi_i^\dagger \varphi_j^\dagger + h.c.) = \frac{1}{2}\Psi^\dagger \Gamma \Psi + \frac{1}{2}TrA, \tag{1.2}$$

$$\Gamma = \begin{pmatrix} A & B \\ B^* & -A^* \end{pmatrix}, \tag{1.3}$$

$$\Psi = (\varphi_1, \varphi_2, ..., \varphi_1^\dagger, \varphi_2^\dagger, ...)^T, \{\Psi_i, \Psi_j\} = \delta_{ij}. \tag{1.4}$$

Next we assume a unitary S which has the form:

$$S = \begin{pmatrix} U & V^* \\ V & U^* \end{pmatrix}, S^\dagger \Gamma S = \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}, E = diag(\epsilon_1, \epsilon_2, ...) \tag{1.5}$$

The diagonal basis is

$$\Phi = S^{-1}\Psi = (\phi_1, \phi_2, ..., \phi_1^\dagger, \phi_2^\dagger, ...)^T, \{\Phi_a, \Phi_b^\dagger\} = \delta_{a,b} \tag{1.6}$$

in terms of which Γ becomes

$$\Gamma = \frac{1}{2}\sum_{a=1}^{N} \epsilon_a(\phi_a^\dagger \phi_a - \phi_a \phi_a^\dagger) + \frac{1}{2}TrA = \sum_{a=1}^{N} \epsilon_a \phi_a^\dagger \phi_a + \frac{1}{2}(TrA - \sum_{a=1}^{N} \epsilon_a) \tag{1.7}$$

The density matrix in the diagonal basis can be factorized,

$$\rho = \Pi_a \frac{\rho_a}{Z_a}, \rho_a = \exp(-\varphi_a \phi_a^\dagger \phi_a), Z_a = Tr\rho_a = 1 + e^{-\varphi_a}. \tag{1.8}$$

With factorization, one can easily verify that

$$\langle \phi_a \phi_b^\dagger \rangle = \frac{\delta_{a,b}}{e^{-\varphi_a} + 1}, \langle \phi_a \phi_b \rangle = \langle \phi_a^\dagger \phi_b^\dagger \rangle = 0 \tag{1.9}$$

The correlation matrix in the Φ basis is defined as (note that $\Phi\Phi^\dagger$ is a matrix ofoperators)

$$G^\Phi = \langle \Phi\Phi^\dagger \rangle = \begin{pmatrix} \frac{1}{e^{-E}+1} & 0 \\ 0 & \frac{1}{e^{+E}+1} \end{pmatrix} = \frac{1}{e^{-S^\dagger \Gamma S}+1} \tag{1.10}$$

which is diagonal. Transforming back to the Ψ basis yields

$$G^\Psi = \langle \Psi\Psi^\dagger \rangle = SG^\Phi S^\dagger = \frac{1}{\exp(-\Gamma)+1} = 1 - \frac{1}{\exp(\Gamma)+1} \tag{1.11}$$

and the inverse relation

$$\Gamma = \ln \frac{G^\Psi}{1-G^\Psi} \tag{1.12}$$

Define the $\rho_a^n = \frac{\rho_a}{Z_a}$ as the normalized density matrix for the diagonal modes. Then the total entropy is

$$S = -Tr[\rho \ln \rho] = -Tr[(\Pi_a \rho_a^n)\ln(\Pi_a \rho_a^n)] = -\sum_a Tr[\rho_a^n \ln \rho_a^n] = \sum_a S_a \tag{1.13}$$

$$S_a = -Tr[\rho_a^n \ln \rho_a^n] = Tr[\frac{e^{-\epsilon_a \phi_a^\dagger \phi_a}}{\ln} \frac{e^{-\epsilon_a \phi_a^\dagger \phi_a}}{1+e^{-\epsilon_a}}] = \frac{1}{1+e^{-\epsilon_a}} \ln \frac{1}{1+e^{-\epsilon_a}} + \frac{e^{-\epsilon_a}}{1+e^{-\epsilon_a}} \ln \frac{e^{-\epsilon_a}}{1+e^{-\epsilon_a}} \tag{1.14}$$

$$= -f_a \ln f_a - (1-f_a)\ln(1-f_a) \tag{1.15}$$

$$f_a = \frac{1}{1+e^{-\epsilon_a}} \tag{1.16}$$

1.2 Replica Method

Consider a relativistic field theory defined on a 1+1 space-time without boundaries. Suppose that the theory is in a thermal state with temperature $1/\beta$. This state is represented by the density matrix $\rho = \exp(?\beta H)/Tr(\exp(-\beta H))$ where H is the Hamiltonian of the theory:

$$\rho(\phi(x^{''})|\phi(x^{'})) = Z^{-1}\langle\phi(x^{''})|\exp(-\beta H)|\phi(x^{'})\rangle \tag{1.17}$$

Observe that ρ is similar to the time evolution operator $\exp(-itH)$ after performing a Wick rotation it $it \to \tau$. The time evolution operator gives the propagator of the theory, the probability amplitude that the system evolves from a particular state to another one after a time interval t. In the path integral representation, this probability amplitude is expressed as the integral over all the possible configurations of the fields that connect the initial and the final state. Then each entry of ρ may be written as a path integral defined on the Euclidean space-time strip of width β represented in Fig. 1.1, connecting a particular configuration at $\tau = 0$ with another one at $\tau = \beta$:

$$\rho(\phi(x^{''})|\phi(x^{'})) = Z^{-1}\int[D\phi(x,\tau)]\prod_x \delta(\phi(x,0)-\phi(x^{'}))\delta(\phi(x,\beta)-\phi(x^{''}))e^{-S_E} \tag{1.18}$$

where $S_E = \int_0^\beta L_E d\tau$ with L_E the Euclidean Lagrangian.

The trace of $\exp(-\beta H)$ is performed by setting the same initial and final configuration and integrating over all the possible states. This is equivalent to the path integral over the cylinder of circumference of length β obtained by gluing the edges of the strip at $\tau = 0$ and $\tau = \beta$.

Now let us consider in the spacial dimension a set X of P=1 disjoint intervals $X = [u, v]$, where u, v denotes the end point of the interval. In Fig. 1.1, the interval of X corresponds to the segments depicted at the edges of the stripe at $\tau = 0, \beta$.

In order to compute the entanglement entropy of these intervals we need the reduced density matrix $Tr_{\overline{X}}\rho$. To compute this partial trace we have to set equal the configuration of the fields at $\tau = 0$ and $\tau = \beta$ at the points of the space that are not in X. In the path integral representation, this corresponds to joining together the edges of the strip at $\tau = 0$ and $\tau = \beta$ except at the points that belong to X. Then we obtain a cylinder like that in Fig. 1.1, with open cuts in the intervals $[u, v]$ that form the subsystem X.

From the thermal state one can recover the ground state $\rho = |GS\rangle\langle GS|$ taking the zero temperature limit $\beta \to \infty$. In this limit the radius of the cylinder goes to infinity. Then the path integrals that give the entries of $\rho_X = Tr_{\overline{X}}|GS\rangle\langle GS|$ are defined on a plane similar to the one represented in Fig. 1.1, with cuts along the segments $[u, v]$ corresponding to the intervals of X.

Now an integer power ρ_X^n can be computed with the replica trick. It consists in taking n copies of the path integrals that represent ρ_X and combining them as follows. Each copy is defined on a plane with P cuts like that of Fig. 1.1. We paste them together along the open cuts $[u, v]$ as we illustrate in Fig. 1.1. That is, if we go around the endpoints u clockwise we move to the upper copy while going around the points v clockwise we move to the lower one. Finally, the trace of ρ_X^n is obtained by joining the first and the last copies. This produces an n-sheeted Riemann surface with branch points at the endpoints u, v of the intervals of X. Then

$$Tr\rho_X^n = \frac{Z_n}{Z_1^n} \tag{1.19}$$

Since $Tr\rho_X^n = \sum_\lambda \lambda^n$, where λ are eigenvalues of ρ_X, the unique analytic contination gives

$$S_X = -\sum_\lambda \lambda \ln \lambda = -(\frac{\partial}{\partial n}\sum_\lambda \lambda^n)|_{n\to 1} = -(\frac{\partial}{\partial n}Tr\rho_X^n)|_{n\to 1} = -(\frac{\partial}{\partial n}\frac{Z_n}{Z_1^n})|_{n\to 1} \tag{1.20}$$

The discussion on the replica trick is valid for all the theories, including the critical and non-critical ones.

In conclusion, reduced density matrix is given by the path integral over the compact Riemann surface of Fig. that can be identified with the partition function of the field theory defined on this surface.

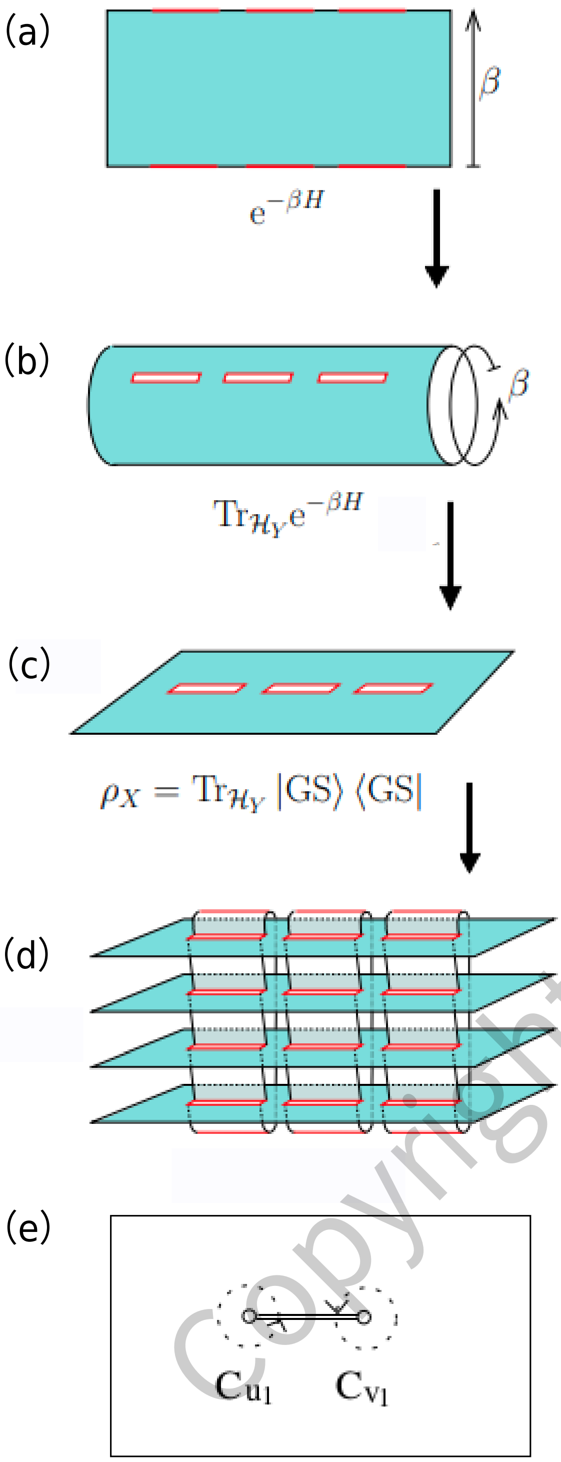

Figure 1.1: The entries of the thermal density matrix $\exp(-\beta H)$ can be represented as path integrals defined on the strip of the Euclidean space-time of width β represented in (a). If we consider a set $X = [u_i, v_i]$ of disjoint intervals in the real space (the red segments), the reduced density matrix $Tr_{\overline{X}} \exp(-\beta H)$ is represented by a cylinder of circumference β with open cuts along the intervals of X, like that in (b). In the limit $\beta \to \infty$ we get the ground state and the corresponding reduced density matrix ρ_X can be seen as a path integral on the plane with open cuts at the intervals of X. Then the quantity $Z_X^n = Tr(\rho_X^n)$ can be interpreted as the partition function of the theory defined on the compact Riemann surface obtained by taking n copies of the plane with cuts represented in (c) and pasting them cyclically along the cuts as it is described in (d).

1.3 1+1d Free Boson Field Theory

We perform an explicit calculation for a free boson theory (Gaussian model). The action is

$$\mathcal{S} = \int d^2x[(\partial_\mu\varphi)^2 + m^2\varphi^2]. \tag{1.21}$$

Next we consider a n-sheeted Riemann surface with one cut, which we arbitrarily fix on the real negative axis. We need to know the ratio Z_n/Z_1^n, where Z_n is partition fuction in the n-sheeted geometry. There are several equivalent ways to calculate partition function. The following way may be the simplest one, by using the quantity:

$$\frac{\partial}{\partial m^2}\ln Z_n = -\int d^2xG_n(\mathbf{x},\mathbf{x}') \tag{1.22}$$

where $G_n(\mathbf{x},\mathbf{x}')$ is the two-point correlation function in the n-sheeted geometry. Thus we need the combination $G_n - nG_1$. G_n obeys

$$(-\nabla^2 + m^2)G_n(\mathbf{x},\mathbf{x}') = \delta^2(\mathbf{x},\mathbf{x}') \tag{1.23}$$

We need to solve it first. In general, Green function can be solved if the eigenstates are known.

1.3.1 Green's function from the eigenvalue expansion

First, let us recal the eigenvalue problem for 2D Helmholtz equation:

$$\nabla^2 u = -\lambda u, u = R(\rho)\Phi(\varphi) \tag{1.24}$$

$$\frac{\rho}{R(\rho)}\frac{d}{d\rho}[\rho\frac{dR(\rho)}{d\rho}] + \frac{1}{\Phi(\varphi)}\frac{d^2\Phi(\varphi)}{d\varphi^2} + \lambda\rho^2 = 0 \tag{1.25}$$

$$\to \nu^2 = -\frac{1}{\Phi(\varphi)}\frac{d^2\Phi(\varphi)}{d\varphi^2} = \frac{\rho}{R(\rho)}\frac{d}{d\rho}[\rho\frac{dR(\rho)}{d\rho}] + \lambda\rho^2 \tag{1.26}$$

The solution to $\Phi(\varphi)$ is

$$\Phi(\varphi) = A_\nu\cos(\nu\varphi) + B_\nu\sin(\nu\varphi)\, or\; A_\nu e^{i\nu\varphi}, \tag{1.27}$$

The solution to radial part is

$$\frac{\rho}{R(\rho)}\frac{d}{d\rho}[\rho\frac{dR(\rho)}{d\rho}]+\lambda\rho^2-\nu^2=0 \tag{1.28}$$

$$\rho^2R^{''}(\rho)+\rho R^{'}(\rho)+(\lambda\rho^2-\nu^2)R(\rho)=0 \tag{1.29}$$

By setting $x=\sqrt{\lambda}\rho$, we have $x^2y^{''}(x)+xy^{'}(x)+(x^2-\nu^2)y(x)=0$, and its solution is Bessel function $y(x)=AJ_\nu(x)+BN_\nu(x)$.

If we require the solution is regular at $\rho=0$, we need discard the solution $N_\nu(x)$ which is divergent at $x=0$. Thus the complete set of eigenfunction is

$$\phi_\nu(\mathbf{r})=N_\nu e^{i\nu\theta}J_\nu(\sqrt{\lambda}\rho),\quad \nu=...,-1,0,1,2,... \tag{1.30}$$

Eigenvalue λ and normalization factor N_ν can be determined in sequence. We consider a finite disk and boundary condition $\phi_\nu(\rho=L)=0$, leading to

$$\sqrt{\lambda}L=\alpha_{\nu,i},\rightarrow\lambda_{\nu,i}=\alpha^2_{\nu,i}/L^2 \tag{1.31}$$

where $\alpha_{\nu,i}$ is i-th zero of the Bessel function $J_\nu(x)$.

The normalization factor is determined by

$$\int d\mathbf{r}\phi^*_\nu(\mathbf{r})\phi_{\nu'}(\mathbf{r})=\delta_{\nu,\nu'}, \tag{1.32}$$

$$\rightarrow N^2_\nu\int_0^{2\pi}d\theta\int_0^L drrJ^2_\nu(\alpha_{\nu,i}r/L)=N^2_\nu 2\pi\frac{L^2}{2}J^2_{\nu+1}(\alpha_{\nu,i}) \tag{1.33}$$

$$N^2_\nu=\frac{1}{2\pi}\frac{2/L^2}{J^2_{\nu+1}(\alpha_{\nu,i})} \tag{1.34}$$

Alternatively, we find the asymptotic form of normalization factor N_v is very helpful in the following calculation. That is,

$$N^2_\nu\int_0^{2\pi}d\theta\int_0^L drr[J^2_\nu(\alpha_{\nu,i}r/L)]\sim N^2_\nu 2\pi\int_0^L drr[\frac{2}{\pi\lambda r}\cos^2(\lambda r-\frac{\nu\pi}{2}-\frac{\pi}{4})] \tag{1.35}$$

$$=N^2_\nu 2\pi\frac{2}{\pi\lambda}\frac{L}{2}=2N^2_\nu L/\lambda_{\nu,i}=1\Rightarrow N^2_\nu=\frac{\lambda_{\nu,i}}{2L} \tag{1.36}$$

Then we assume the form of green function as $G(\mathbf{x},\mathbf{x}')=\sum_\nu C_\nu\phi_\nu(\mathbf{x})$, and insert it into the equation $(-\nabla^2 + m^2)G(\mathbf{x}, \mathbf{x}') = \delta^2(\mathbf{x}, \mathbf{x}')$,

$$-\sum_\nu C_\nu(-\lambda_{\nu,i})\phi_\nu(\mathbf{r}) + m^2 \sum_\nu C_\nu \phi_\nu(\mathbf{r}) = \delta(\mathbf{r} - \mathbf{r}') \rightarrow \qquad C_\nu = \frac{1}{\lambda_{\nu,i} + m^2}\phi_\nu(\mathbf{r}') \tag{1.37}$$

Finally, we have

$$G(\mathbf{r}, \mathbf{r}') = \sum_\nu N_v^2 \frac{J_\nu(\sqrt{\lambda_{\nu,i}}r)J_\nu(\sqrt{\lambda_{\nu,i}}r')}{\lambda_{\nu,i} + m^2} e^{i\nu(\theta-\theta')} \tag{1.38}$$

$$= \frac{1}{2\pi} \sum_\nu \sum_i \frac{2/L^2}{J_{\nu+1}^2(\alpha_{\nu,i})} \frac{J_\nu(\sqrt{\lambda_{\nu,i}}r)J_\nu(\sqrt{\lambda_{\nu,i}}r')}{\lambda_{\nu,i} + m^2} e^{i\nu(\theta-\theta')} \tag{1.39}$$

In the above formulas, we just use a 1-sheeted geometry $n = 1$. To imposing the $2\pi n$ periodicity boundary condition on n-sheeted geometry, we take $\nu = k/n$. This will modify the wave function set and Green function, $N_\nu^2 = \frac{1}{2\pi n} \frac{2/L^2}{J_{\nu+1}^2(\alpha_{\nu,i})}$.

$$G_n(\mathbf{r}, \mathbf{r}') = \sum_\nu N_v^2 \frac{J_\nu(\sqrt{\lambda_{\nu,i}}r)J_\nu(\sqrt{\lambda_{\nu,i}}r')}{\lambda_{\nu,i} + m^2} e^{i\nu(\theta-\theta')} \tag{1.40}$$

$$= \frac{1}{2\pi n} \sum_\nu \sum_i \frac{2/L^2}{J_{\nu+1}^2(\alpha_{\nu,i})} \frac{J_\nu(\sqrt{\lambda_{\nu,i}}r)J_\nu(\sqrt{\lambda_{\nu,i}}r')}{\lambda_{\nu,i} + m^2} e^{i\nu(\theta-\theta')} \tag{1.41}$$

$$\tag{1.42}$$

In addition, in the limit of $L \rightarrow \infty$, $\alpha_{\nu,i}/L \rightarrow \lambda$ becomes continuous, and $N_v^2 \rightarrow \frac{\lambda}{2Ln}$. Then $(\sum_\lambda \rightarrow L/\pi \int d\lambda)$

$$G_n(\mathbf{r}, \mathbf{r}') = \sum_\nu N_v^2 \frac{J_\nu(\sqrt{\lambda_{\nu,i}}r)J_\nu(\sqrt{\lambda_{\nu,i}}r')}{\lambda_{\nu,i} + m^2} e^{i\nu(\theta-\theta')} \tag{1.43}$$

$$= \frac{1}{2\pi n} \sum_\nu \int_0^\infty \lambda d\lambda \frac{J_\nu(\lambda r)J_\nu(\lambda r')}{\lambda^2 + m^2} e^{i\nu(\theta-\theta')} \tag{1.44}$$

$$G_n(r, r', \theta = \theta') = \frac{1}{2\pi n} \sum_{\nu \geq 0} d_\nu \int_0^\infty \lambda d\lambda \frac{J_\nu(\lambda r)J_\nu(\lambda r')}{\lambda^2 + m^2}, \tag{1.45}$$

where $d_{\nu=0} = 1, d_{\nu>0} = 2$ (please note this relation from sum of ν).

1.3.2 Partition function

Next we perform integeral on λ and θ (from 0 to $2\pi n$), then we obtain

$$G_n(r, r') = \sum_{\nu \geq 0} d_\nu I_\nu(mr) K_\nu(mr') = \sum_{k \geq 0} d_k I_{k/n}(mr) K_{k/n}(mr') \tag{1.46}$$

where $I_n(x), K_n(x)$ is imaginary Bessel functions, and we used the relation $\int_0^\infty J_\nu(at) J_\nu(bt) \frac{tdt}{t^2+c^2} = I_\nu(bz) K_\nu(az), (a > b)$.

Next we use the above results to calculate the partition function as

$$-\frac{\partial}{\partial m^2} \ln Z_n = \int dr^2 G_n(r, r) = \int r dr [\sum_{k \geq 0} d_k I_{k/n}(mr) K_{k/n}(mr)] \tag{1.47}$$

To proceed it, we need the help of the Euler-MacLaurin sum formula,

$$\frac{1}{2} \sum_{k=0}^{\infty} d_k f(k) = \int_0^\infty f(k) dk - \frac{1}{12} f^{'}(0) - \sum_{j=2}^{\infty} \frac{B_{2j}}{(2j)!} f^{(2j-1)}(0), \tag{1.48}$$

where B_{2n} are the Bernoulli numbers. By introducing a regularized function $F(k/(n\Lambda))$ (which goes to zero fast enough and takes $F(0) = 1$), we have

$$\sum_{k \geq 0} d_k I_{k/n}(mr) K_{k/n}(mr) = \sum_{k=0}^{\infty} d_k I_{k/n}(mr) K_{k/n}(mr) F(k/(n\Lambda)) \tag{1.49}$$

$$= 2 \int_0^\infty dk I_{k/n}(mr) K_{k/n}(mr) F(k/n\Lambda) + \frac{1}{6n} K_0^2(mr) - \int r dr \sum_{j \geq 1} \frac{B_{2j}}{(2j)!} D_{(2j+1)}(0) \tag{1.50}$$

with $\partial_k K_k(x)|_{k=0} = 0, \partial_k I_k(x)|_{k=0} = -K_0(x)$, and $D_j(x) = \partial^i (I_k(x) K_k(x)) / \partial k^i |_{k=0}$. In the last term the integral, derivative and sum can be exchanged and each term in the sum is

$$\frac{\partial^i}{\partial k^i} (\int x dx I_k(x) K_k(x)) = -\frac{\partial^i}{\partial k^i} \frac{k}{2} = 0, i = (2j+1) \geq 2 \tag{1.51}$$

Inserting the results into the partition function leads to

$$\frac{\partial}{\partial m^2} \ln Z_n = -2 \int_0^\infty r dr \int_0^\infty dk I_{k/n}(mr) K_{k/n}(mr) F(k/n\Lambda) - \frac{1}{12nm^2} \tag{1.52}$$

where we used $\int_0^\infty r dr K_0^2(mr) = 1/(2m^2)$.

Notice that the integeral over k can be rescaled by $k \to nk$, as

$$\frac{\partial}{\partial m^2} \ln Z_n = -2n \int_0^\infty rdr \int_0^\infty dk I_k(mr) K_k(mr) F(k\Lambda) - \frac{1}{12nm^2} \tag{1.53}$$

Thus,

$$\frac{\partial}{\partial m^2} \ln Z_n - n \frac{\partial}{\partial m^2} \ln Z_1 = \frac{\partial}{\partial m^2} \ln \frac{Z_n}{Z_1^n} = -\frac{1}{12nm^2} + \frac{n}{12m^2} \tag{1.54}$$

$$\to \ln Tr\rho^n = \ln \frac{Z_n}{Z_1^n} = \frac{\ln m^2}{12}[n - \frac{1}{n}], \tag{1.55}$$

The entanglement entropy is obtained by

$$S = -Tr[\rho \ln \rho] = -\frac{\partial}{\partial n} Tr\rho^n|_{n=1} = -\frac{\partial}{\partial n}[m^2]^{\frac{1}{12}(n-\frac{1}{n})}|_{n=1} = -\frac{1}{6} \ln m^2 = \frac{1}{3} \ln \xi \tag{1.56}$$

which gives central charge $c = 1$ by comparing $S = \frac{c}{3} \ln \xi$.

At last, several remarks set in sequence. If we go back to Eq. 1.47, we see an important thing: The integral and sum can not exchange orders. If the order is exchanged, we got

$$-\frac{\partial}{\partial m^2} \ln Z_n = \int dr^2 G_n(r,r) = \int rdr[\sum_{k\geq 0} d_k I_{k/n}(mr) K_{k/n}(mr)] \tag{1.57}$$

$$\stackrel{!}{=} \sum_{k\geq 0} d_k \int rdr I_{k/n}(mr) K_{k/n}(mr) = \frac{1}{2nm^2} \sum_{k\geq 0} d_k k \tag{1.58}$$

The sum is UV divergent. But if making a detour, connecting it to the zeta function $\zeta(s) = \sum_{s=1}^{\infty} \frac{1}{n^s}$, we get $\sum_{k\geq 0} kd_k = 2\zeta(-1)$. What is the value of $\zeta(-1)$? After analytical continuation (to define ζ well define on the whole complex plane), we know $\zeta(-1) = -\frac{1}{12}$. In this regarding, we have again $\frac{\partial}{\partial m^2} \ln \frac{Z_n}{Z_1^n} = -\frac{1}{12nm^2} + \frac{n}{12m^2}$. It is the same with the correct result, but not in correct integral and sum order! This is really interesting, because: 1) this is one novel example integral and sum can not exchange order arbitrarily; 2) we connect to Riemann zeta function in physics!