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Chapter 1

Zero temperature Green's function in Physics

The method of Green's functions allows to formulate specific models of interacting particles, as well as constitutes the universal method of practical calculations of arbitrary physical properties of many particle systems with the account of different types of interactions. This method originated in quantum field theory, which is quite effective and convenient approach, based on the use of Feynman diagrams. The following transfer of these methods to the theory of many particle systems, in fact, lead to the formulation of the modern theory of condensed matter physics.

The general theory developed in previous chapter can be applied directly to the time-independent and time-dependent one-particle Schrodinger equation by making the substitutions $L(r) \to H(r), z \to E$, where $H(r)$ is the Hamiltonian. That are from formally mathematical point of view. In this chapter, we will discuss the definition of Green's function from the point of view of physics. Then we will discuss analytical properties of Greens functions and their relation to quasiparticles. Finally, we will present perturbation theory and diagram techniques for Greens functions at zero temperature.

1.1 A Warm up for Green's function

Consider first the case of temperature $T = 0$, i.e. the system is at its ground state. Let us start from the elementary problem of a single quantum particle moving in some time-independent external potential (or field), and described by the usual (time-dependent) Schrodinger equation with appropriate Hamiltonian H:

$$i\frac{\partial\psi(\mathbf{r},t)}{\partial t} - H\psi(\mathbf{r},t) = 0, or \quad i\frac{\partial|\psi(t)\rangle}{\partial t} - H|\psi(t)\rangle = 0 \tag{1.1}$$

Instead of solving this equation directly (with some initial condition for the wave-function) we introduce the Schrodinger-like equation for the Green's function, depending on two values of time and coordinate:

$$i\frac{\partial G(\mathbf{r},t;\mathbf{r}',t')}{\partial t} - HG = i\delta(\mathbf{r}-\mathbf{r}')\delta(t-t'), or \quad i\frac{\partial G(t,t')}{\partial t} - HG = i\delta(t-t') \tag{1.2}$$

with initial condition $G(\mathbf{r},t;\mathbf{r}',t') = \delta(\mathbf{r}-\mathbf{r}')$. Physically, Green's function represents the probability amplitude for a particle transition from (initial) point $\mathbf{r}'$ at the moment of time t' to the some point $\mathbf{r}$ at the moment t. This is easily checked expressing wave-function at the moment t via wave-function at the moment t' as $\psi(r,t) = \int dr'dt'G(r,t;r',t')\psi(r',t')$ or $|\psi(t)\rangle = G(t,t')|\psi(t')\rangle$.

Next, we would like to introduce the concept of 'retarded' and 'advanced', which respectively relates to the time evolution along the positive time arrow $t > t'$ and time along negative time arrow $t < t'$. That is, $G^R(t) = G^+(t) = 0$ for $t < 0$. Next most of discussion is on the retarded green's function.

Let us now introduce some set of eigenfunctions of the stationary Schrodinger equation (H doesnot depend on t):

$$H\varphi_\lambda(r) = E_\lambda\varphi_\lambda(r), or, H|\lambda\rangle = E_\lambda|\lambda\rangle \tag{1.3}$$

Depending on the problem at hand, the quantum numbers λ can have different physical meaning. Any solution of the Schrodinger equation can be expanded using the complete system of

eigenfunctions:

$$\psi(r,t) = \sum_\lambda c_\lambda(t)\varphi_\lambda(r), or|\psi(t)\rangle = \sum_\lambda c_\lambda(t)|\lambda\rangle \tag{1.4}$$

then we have (please recall Eq. **??**)

$$|\psi(t)\rangle = \sum_\lambda c_\lambda(t)|\lambda\rangle = iG^R(t,t')|\psi(t')\rangle = iG^R(t,t')\sum_{\lambda'} c_{\lambda'}(t')|\lambda'\rangle \Rightarrow \tag{1.5}$$

$$c_\lambda(t) = \sum_{\lambda'} iG^R_{\lambda\lambda'}(t,t')c_{\lambda'}(t), G^R_{\lambda\lambda'}(t,t') = \langle\lambda|G^R(t,t')|\lambda'\rangle \tag{1.6}$$

Here $G^R_{\lambda\lambda'}(t,t')$ - the Green's function in the representation of quantum numbers λ. As $|\lambda\rangle$ is an exact stationary state of the (time-independent) Hamiltonian H, there are no transitions to another states, so that (from time-evolution operator: $e^{-iH(t-t')}|\psi(t')\rangle$) $c_\lambda(t) = e^{-iE_\lambda(t-t')}c_\lambda(t')$, i.e.

$$G^R_{\lambda\lambda'}(t,t') = G^R_\lambda(t-t')\delta_{\lambda,\lambda'} = -ie^{-iE_\lambda(t-t')}\theta(t-t') \tag{1.7}$$

where $\theta(t-t') = 1$ for $t > t'$ and $\theta(t-t') = 0$ for $t < t'$. Here the condition $\theta(t-t')$ indicates the nature of retarded green function.

Consider the Fourier transformation:

$$G^R_\lambda(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t}G^R_\lambda(t) \tag{1.8}$$

$$G^R_\lambda(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t}G^R_\lambda(\omega) \tag{1.9}$$

After elementary integration we get

$$G^R_\lambda(\omega) = \frac{1}{\omega + i\eta - E_\lambda}, \eta = 0^+ \tag{1.10}$$

$$G^R_\lambda(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t}\frac{1}{\omega + i\eta - E_\lambda} = \begin{cases} -ie^{-iE_\lambda t}e^{-\eta t}, t > 0 \\ 0, t < 0 \end{cases} \tag{1.11}$$

To convince yourself note, that the integral here has a pole at $\omega = E_\lambda - i\eta$. Then for $t > 0$ we can close the integration contour in the lower half-plane of complex variable ω (as the factor $e^{-i\omega t} = e^{-i(Re\omega)t+(Im\omega)t}$ guarantees the exponential damping of the integral at the semicircle at

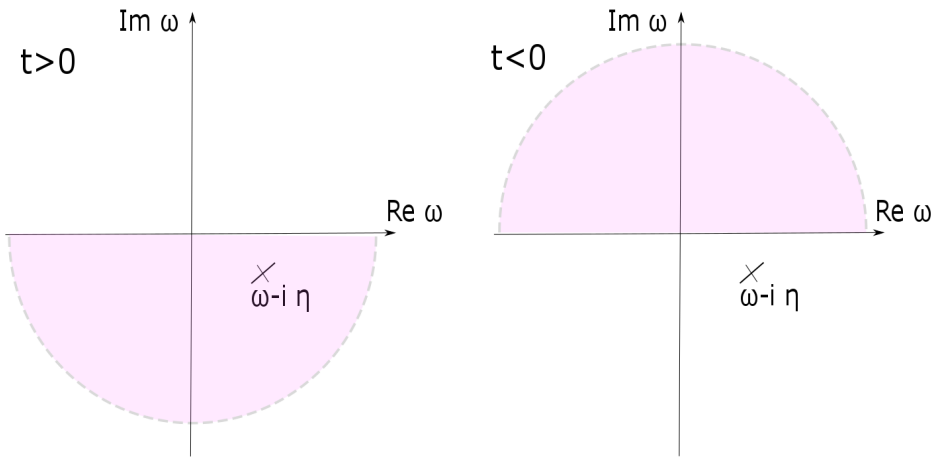


Figure 1.1: Integral contour in complex plane.

infinity in the lower half-plane), then the pole of the integral is inside the contour of integration and Cauchy theorem can be used. For $t < 0$, to guarantee the zero contribution from the semicircle, we have to close integration contour in the upper half-plane of ω. Then there is no pole inside the contour and the integral reduces to zero.

Here we prove the relation below ($\eta = 0^+$):

$$-i\int_{-\infty}^{\infty} dt e^{i\omega t} e^{-ixt}\theta(t) = \frac{1}{\omega + i\eta - x} \tag{1.12}$$

Let us make the integral directly:

$$\begin{aligned} -i\int_{-\infty}^{\infty} dt e^{i\omega t} e^{-ixt}\theta(t) &= -i\int_{0}^{\infty} dt e^{i\omega t} e^{-ixt} \\ &= -\frac{1}{\omega + i\eta - x}\int_{0}^{\infty} [d(\omega + i\eta - x)it] e^{i(\omega + i\eta - x)t} \\ &= -\frac{1}{\omega + i\eta - x}[e^{i(\omega + i\eta - x)t}]|_0^{\infty} = \frac{1}{\omega + i\eta - x} \end{aligned} \tag{1.13}$$

Here we prove the relation below($\eta = 0^+$):

$$\theta(t)e^{-ixt} = i\int \frac{d\omega}{2\pi}\frac{e^{-i\omega t}}{\omega + i\eta - x} \tag{1.14}$$

i) $t > 0$ case, we choose the integral contour in the lower half-plane of complex plane

with $Im\omega < 0$ (as the factor $e^{-i\omega t} = e^{-i(Re\omega)t+(Im\omega)t}$ guarantees the exponential damping $e^{(Im\omega)t} \to 0$ of the integral at the semicircle at infinity in the lower half-plane). Please see Fig. 1.1 for details. Then we apply the residue theorem: $\int dz f(z) = 2\pi i Res[f(z)(z-z_0)]|_{z\to z_0}$:

$$i\int \frac{d\omega}{2\pi}\frac{e^{-i\omega t}}{\omega + i\eta - x} = (-1)\times\frac{i}{2\pi}(2\pi i)e^{-i(x-i\eta)t} \to e^{-ixt} \tag{1.15}$$

where (-1) is from the integral direction of the closed loop.

ii) $t < 0$ case, we choose the integral contour in the upper half of complex plane with $Im\omega > 0$ ($e^{-i\omega t} = e^{-i(Re\omega)t+(Im\omega)t} \to 0$). But, in this case, there is no poles enclosed by the contour, so the integral should vanish.

Now consider the many-particle system. Let us limit discussion only to the case of (many) fermions. Let us determine Green's function for a particle excitation $G_{\lambda\lambda'}(t)$, i.e. the transition amplitude of a particle from some state λ to a state λ' (for the case of non-interacting Fermions). We have to take into account limitations due to Pauli principle, i.e. exclude transitions to occupied states. This can be achieved by an additional factor $(1 - n_\lambda)$ in the definition of the Green's function, where

$$n_\lambda = \begin{cases} 1, E_\lambda \le E_F = 0 \\ 0, E_\lambda > E_F = 0 \end{cases} \tag{1.16}$$

is the particle number in a state λ (again $T = 0$). Thus we obtain the electron Green's function

$$G^e_{\lambda\lambda'}(t) \sim (1-n_\lambda)\delta_{\lambda\lambda'}\begin{cases} -ie^{-iE_\lambda t}, t > 0 \\ 0, t < 0 \end{cases} \tag{1.17}$$

and the hole Green's function

$$G^h_{\lambda\lambda'}(t) \sim n_\lambda\delta_{\lambda\lambda'}\begin{cases} -ie^{-iE_\lambda t}, t > 0 \\ 0, t < 0 \end{cases} \tag{1.18}$$

where E_λ accounts from the Fermi level E_F.

Based on this observation, we now consider the expression of the Green's function in terms

of the operators, by going to the second quantization picture:

$$electron : G^e_\lambda(t,t') = -i\langle 0|\hat{c}_\lambda(t)\hat{c}^\dagger_\lambda(t')|0\rangle = -ie^{-iE_\lambda(t-t')}\langle 0|\hat{c}_\lambda\hat{c}^\dagger_\lambda|0\rangle \tag{1.19}$$

$$hole : G^h_\lambda(t,t') = -i\langle 0|\hat{c}^\dagger_\lambda(t')\hat{c}_\lambda(t)|0\rangle = -ie^{iE_\lambda(t-t')}\langle 0|\hat{c}^\dagger_\lambda\hat{c}_\lambda|0\rangle \tag{1.20}$$

where it is taken into account that annihilation of a particle in a given point is equivalent to creation of a hole. Both of above expressions are defined for $t > t'$. Alternatively, we can write everything in a single concrete form:

$$G_\lambda(t;t') = \begin{cases} G^e_\lambda(t;t'), t > t' \\ -G^h_\lambda(t';t), t < t' \end{cases} \Rightarrow G_\lambda(t,t') = -i\langle 0|T\hat{c}_\lambda(t)\hat{c}^\dagger_\lambda(t')|0\rangle \tag{1.21}$$

where the symbol T means all the operators are placed in order over time arguments: $TF_1(t_1)F_2(t_2) = \begin{cases} F_1(t_1)F_2(t_2), t_1 > t_2 \\ -F_2(t_2)F_1(t_1), t_1 < t_2 \end{cases}$. This is the general definition for the many-particle Green's function, and we will discuss it in detail in the following.

For a general case, suppose we have operators $F_1(t_1), ..., F_N(t_N)$ at different times $t_{P_1} > t_{P_2} > .. > t_{P_N}$, the time-ordering operator is defined as:

$$T[F_1(t_1), ..., F_N(t_N)] = \begin{cases} (-)^P F_{P_1}(t_{P_1}), ..., F_{P_N}(t_{P_N}), Fermion \\ F_{P_1}(t_{P_1}), ..., F_{P_N}(t_{P_N}), Boson \end{cases} \tag{1.22}$$

, where P is the number of pairwise permutations involved in the time-ordering process.

Next we discuss the physical meaning of Green's function. Please note that, the "many-body" means that we use the many-body ground state instead of vacuum to calculate the average (see below). First, In this definition, we see $t > t'$, Green's function is the inner product by $c^+(t')|0\rangle$ and $c^+(t)|0\rangle$. This describes the amplitude of a state with $N+1$ electrons propragating from t to t', i.e. electron information is encoded. By contrast, for $t < t'$, the green function is the inner product of $c(t)|0\rangle$ and $c(t')|0\rangle$, which encloses $N-1$ electrons. It is the hole propragation amplitude.

At last, we make the Fourier transformation in the frequency space:

$$G_\lambda(t) = \int \frac{d\omega}{2\pi} G_\lambda(\omega) e^{-i\omega t} \tag{1.23}$$

$$G_\lambda(\omega) = \int dt G_\lambda(t) e^{i\omega t} \tag{1.24}$$

And next we will see $G_\lambda(\omega)$ contains more information and is very convenient. So we will mainly focus on $G_\lambda(\omega)$ in this class. The explicit transformation of the Green's function $G(t)$ is

$$G_\lambda(t) = \begin{cases} G^e_\lambda(t), t > 0 \\ -G^h_\lambda(-t), t < 0 \end{cases} \tag{1.25}$$

$$\begin{aligned} \Rightarrow G_\lambda(\omega) &= -i(1-n_\lambda)\int_0^\infty dt e^{-iE_\lambda t + i\omega t} + in_\lambda \int_{-\infty}^0 dt e^{iE_\lambda t + i\omega t} \\ &= \frac{1-n_\lambda}{\omega + i\eta - E_\lambda} + \frac{n_\lambda}{\omega - i\eta + E_\lambda} \\ &\equiv \frac{1}{\omega + i\eta sign(E_\lambda) - E_\lambda}, \begin{cases} E_\lambda > 0, E_\lambda > E_F \\ E_\lambda < 0, E_\lambda < E_F \end{cases} \end{aligned} \tag{1.26}$$

where $\eta = 0^+$ is necessary to guarantee convergence. The sign function is defined as $sign(x) = 1(-1)$ for $x > 0 (x < 0)$.

1.2 Green's function in physics

With previous preparation in the previous section, let us give the formal definition of many-body Green's function:

$$iG_{ab}(xt, x't') = \frac{\langle \Psi_0 | T[\psi_{Ha}(xt)\psi^\dagger_{Hb}(x't')] | \Psi_0 \rangle}{\langle \Psi_0 | \Psi_0 \rangle} \tag{1.27}$$

where $|\Psi_0\rangle$ is the Heisenberg ground state of the interacting system satisfying $H|\Psi_0\rangle = E|\Psi_0\rangle$, and $\psi_H(xt)$ is a Heisenberg field operator with the time dependence $\psi_{Ha}(xt) = e^{iHt}\psi_a(x)e^{-iHt}$. The time-ordering operator represents $T[\psi_{Ha}(xt)\psi^\dagger_{Hb}(x't')] = \begin{cases} \psi_{Ha}(xt)\psi^\dagger_{Hb}(x't'), t > t' \\ \pm\psi^\dagger_{Hb}(x't')\psi_{Ha}(xt), t < t' \end{cases}$, where

$+(-)$ is for boson (fermion) operator. If we use the definition of time-ordering, we get the form

$$iG_{ab}(xt,x't') = \begin{cases} e^{iE(t-t')}\frac{\langle\Psi_0|\psi_a(x)e^{-iH(t-t')}\psi_b^\dagger(x')]|\Psi_0\rangle}{\langle\Psi_0|\Psi_0\rangle}, t>t' \\ \pm e^{-iE(t-t')}\frac{\langle\Psi_0|\psi_b^\dagger(x')e^{iH(t-t')}\psi_a(x)]|\Psi_0\rangle}{\langle\Psi_0|\Psi_0\rangle}, t<t' \end{cases} \tag{1.28}$$

One can compare this "formal definition" with the "warm up " one in Eq. 1.21. There are several differences, as discussed below. First, the physical meaning of zero temperature Green's function is, the proprability of the process that creating a particle at time t' from the ground state (with N-particles) and then deconstruction this particle at time t. This is the reason why we spent some time to study the "warm-up" section as above.

Second, compared with the single particle case, the definition for many-body Green's function is based on ground state but rather the vaccum: For single-particle case, we use $|0\rangle$ to get the average, while for many-particle case, we use many-body ground state to calculate the average.

Here we only use the ground state to evaluate the average. Generally speaking, for the average of operator $\langle A\rangle$, it is actually

$$\langle A\rangle = \frac{\sum_i\langle i|A|i\rangle e^{-\beta(E_i-\mu N_i)}}{\sum_i e^{-\beta(E_i-\mu N_i)}} \tag{1.29}$$

where $Z = Tr[e^{-\beta(H-\mu N)}] = \sum_i e^{-\beta(E_i-\mu N_i)}$ is the partition function and $\beta = 1/T$ is the inverse of temperature. In the limit of zero temperature, we have $\langle A\rangle = \langle\psi_0|A|\psi_0\rangle$, where $|\psi_0\rangle$. is the ground state. This is the reason to call it as zero-temperature Green's function. The finite temperature case will be discussed in the next chapter.

Third, one big difference is to replace the creation operators by the field operators. For the example of field operator, we know the fermionic operator can be expressed as

$$\hat{\psi}_a(x) = \sum_n c_{n,a}\phi_{n,a}(x), \hat{\psi}_a^\dagger(x) = \sum_n c_{n,a}^\dagger\phi_{n,a}^*(x) \tag{1.30}$$

where $c, c^\dagger$ is creation and destruction operators and $\{\phi_n\}$ is a set of eigenfunctions, e.g. $\phi_k(x) =$

$e^{i\mathbf{k}\cdot\mathbf{x}}/\sqrt{V}$. The field operators satisfy the commutation relation:

$$[\psi_a(x), \psi_b^\dagger(x')]_\zeta = \psi_a(x)\psi_b^\dagger(x') - \zeta\psi_b^\dagger(x')\psi_a(x) = \delta_{ab}\delta(x - x') \tag{1.31}$$

$$[\psi_a(x), \psi_b(x')]_\zeta = [\psi_a^\dagger(x), \psi_b^\dagger(x')]_\zeta = 0 \tag{1.32}$$

where $\zeta = -1(1)$ for fermion (boson). Please note that, the field operator satisfies with $\psi_a|\Psi_0\rangle = 0$, which can be taken as the definition of field operator.

By using the field operators, we are not only limited to the eigenstate space. Instead, we have large freedom to discuss the physics what we have seen in reality, for example, we can work in the spatial space directly. This is a key difference, and please digest it by the readers.

Next, we further discuss some properties of the many-body Green's function.

1.2.1 Lehmann represetation

We start from a more general form of the Green's function, and we insert the complete basis $\sum_n |n\rangle\langle n| = 1$ and get

$$\begin{aligned}
&iG_{ab}(t,t') = \langle 0|T[\psi_a(t)\psi_b^\dagger(t')]|0\rangle \\
&= \sum_n \theta(t-t')\langle 0|\psi_a(t)|n\rangle\langle n|\psi_b^\dagger(t')]|0\rangle - \theta(t'-t)\langle 0|\psi_b^\dagger(t)|n\rangle\langle n|\psi_a(0)|0\rangle \\
&= \sum_n \theta(t-t')e^{-i(E_n-E_0)(t-t')}\langle 0|\psi_a(0)|n\rangle\langle n|\psi_b^\dagger(0)|0\rangle - \theta(t'-t)e^{i(E_n-E_0)(t-t')}\langle 0|\psi_b^\dagger(0)|n\rangle\langle n|\psi_a(0)|0\rangle \\
&\Rightarrow G_{ab}(\omega) = \sum_n \frac{\langle 0|\psi_a(0)|n\rangle\langle n|\psi_b^\dagger(0)|0\rangle}{\omega - (E_n - E_0) + i\eta} + \frac{\langle 0|\psi_b^\dagger(0)|n\rangle\langle n|\psi_a(0)|0\rangle}{\omega + (E_n - E_0) - i\eta}
\end{aligned} \tag{1.33}$$

This is the Lehmann representation for the Green's function. η is introduced for the convergence of the integral (see integral in Eq. 1.12).

If we look at the limit of $\omega \to \infty$, we see

$$G_{ab}(\omega \to \infty) \sim \frac{1}{\omega}\sum_n [\langle 0|\psi_a(0)|n\rangle\langle n|\psi_b^\dagger(0)|0\rangle + \langle 0|\psi_b^\dagger(0)|n\rangle\langle n|\psi_a(0)|0\rangle] = \frac{1}{\omega}\delta_{ab} \tag{1.34}$$

This is a property of Green's function.

Next we can define the retarded and advanced Green's function:

$$iG^R_{ab}(xt,x't') = \theta(t-t')\langle|[\psi_{Ha}(xt),\psi^\dagger_{Hb}(x't')]_\eta|\rangle = \theta(t-t')\langle|\psi_{Ha}(xt)\psi^\dagger_{Hb}(x't') - \zeta\psi^\dagger_{Hb}(x't')\psi_{Ha}(xt)|\rangle \tag{1.35}$$

$$iG^A_{ab}(xt,x't') = \theta(t'-t)\langle|[\psi_{Ha}(xt),\psi^\dagger_{Hb}(x't')]_\eta|\rangle = \theta(t'-t)\langle|\psi_{Ha}(xt)\psi^\dagger_{Hb}(x't') - \zeta\psi^\dagger_{Hb}(x't')\psi_{Ha}(xt)|\rangle \tag{1.36}$$

and its Fourier transformation form:

$$G^R(\omega) = \sum_n [\frac{\langle 0|\psi_a(0)|n\rangle\langle n|\psi^\dagger_b(0)|0\rangle}{\omega-(E_n-E_0)+i\eta} + \frac{\langle 0|\psi^\dagger_b(0)|n\rangle\langle n|\psi_a(0)|0\rangle}{\omega+(E_n-E_0)+i\eta}] \tag{1.37}$$

$$G^A(\omega) = \sum_n [\frac{\langle 0|\psi_a(0)|n\rangle\langle n|\psi^\dagger_b(0)|0\rangle}{\omega-(E_n-E_0)-i\eta} + \frac{\langle 0|\psi^\dagger_b(0)|n\rangle\langle n|\psi_a(0)|0\rangle}{\omega+(E_n-E_0)-i\eta}] \tag{1.38}$$

Here we see $[G^R_{ab}(k,\omega)]^* = G^A_{ba}(k,\omega)$. The retarded and advanced Green's functions differ from each other and from the time-ordered Green's function only in the convergence factors $\pm i\eta$, which are important near the singularities: G^R has singularity in the lower-half complex plane and it is analytic in the upper-half complex plane. As a comparison, G^A has singularities in the upper-half plane. G has singularities in the upper-half complex plane for $\omega<0$, and singularities in the lower-half complex plane for $\omega>0$. We thus conclude another property:

$$\begin{cases} G^R_{ab}(k,\omega) = G_{ab}(k,\omega), \omega>0 \\ G^A_{ab}(k,\omega) = G_{ab}(k,\omega), \omega<0 \\ ReG(k,\omega) = ReG^R(k,\omega) = ReG^A(k,\omega) \\ ImG^R(k,\omega) = sgn(\omega)ImG(k,\omega) = -ImG^A(k,\omega) \end{cases} \tag{1.39}$$

At last, we note that, using the relation $\frac{1}{\omega-\omega_0+i0^+} = P\frac{1}{\omega-\omega_0} - i\pi\delta(\omega-\omega_0)$ and $\frac{1}{\omega-\omega_0} = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{-\pi\delta(\omega'-\omega_0)}{\omega'-\omega}d\omega'$:

$$ReG^R(k,\omega) = \frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{ImG^R(k,\omega')}{\omega'-\omega}d\omega' \tag{1.40}$$

$$ReG^A(k,\omega) = -\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{ImG^A(k,\omega')}{\omega'-\omega}d\omega' \tag{1.41}$$

$$ReG(k,\omega) = \frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{ImG(k,\omega')sgn(\omega')}{\omega'-\omega}d\omega' \tag{1.42}$$

This is the the KramersKronig relation which is satisfied by the real-frequency Green's functions. Thus, the real and imaginary part is not independent.

1.2.2 Spectral function

The spectral function is defined as the imaginary part of the retarded Green's function:

$$A(\omega) = -\frac{1}{\pi} ImG^R(\omega) \tag{1.43}$$

There is one important property for spectral function :

$$\int d\omega A(\omega) = 1 \tag{1.44}$$

This can be proved easily for non-interacting fermion, for example: using Eq. 1.10, we immediately have

$$\int d\omega A(\omega) = \int d\omega \frac{\eta/\pi}{(\omega - E_\lambda)^2 + \eta^2} = \int d\omega \delta(\omega - E_\lambda) = 1 \tag{1.45}$$

The spectral function in this case is just a delta function. The spectral function $A(\omega)$ is interpreted as a probability function. It is the probability that an electron has quantum number λ. For a free, or noninteracting, the probability distribution should be sum up to one.

Or one can also prove this sum-rule using Lehmann represetation:

$$\begin{aligned}\int d\omega A(\omega) &= \int_{-\infty}^{\infty} d\omega \sum_n \langle 0|\psi_a|n\rangle\langle n|\psi_b^+|0\rangle \delta(\omega - (E_n - E_0)) + \sum_n \langle 0|\psi_b^+|n\rangle\langle n|\psi_a|0\rangle \delta(\omega + (E_n - E_0)) \\ &= \langle 0|\psi_a\psi_b^+ + \psi_b^+\psi_a|0\rangle = \delta_{ab}\end{aligned} \tag{1.46}$$

1.2.3 Physical meaning of Green's function

Following the last section, for broadened spectral function, it relates to possess quasi-particles with energy $\tilde{E}$ and a lifetime $1/\eta$. (Particle picture fails at a typical time scale τ) As a simple example we consider a Greens function which decays in time due to processes that scatters the particle out of the state E. In this situation the retarded Greens function becomes (see Eq.

1.15):

$$G^R(t) \approx -i\theta(t)e^{-iEt}e^{-\eta t} = -i\theta(t)e^{-iEt}e^{-t/\tau} \tag{1.47}$$

where τ is the characteristic decay time. Such a decaying Green's function corresponds to a finite width of the spectral function

$$A(\omega) = -Im\int_{-\infty}^{\infty} dte^{i\omega t}G^R(t) = Imi\int_0^{\infty} dte^{i\omega t}e^{-iEt}e^{-t/\tau} = \frac{1/\tau}{(\omega - E)^2 + (1/\tau)^2} \tag{1.48}$$

Here we see that, in the real-time Green's function, the lifetime can be viewed as the decaying factor. In the frequency Green's function, the inverse of lifetime is proportional to broadened width in spectral function. The poles in frequency Green's function is the quasi-particle energy. This lifetime is determined by the imaginary part of pole of Green's function in the frequency domain. It implies that quasi-particle means it doesnot a real particle (which should be stable with infinite lifetime).

The simple notion of single electron propagators becomes less well defined for interacting systems, which is reflected in a broadening of the spectral function. Amazingly, the free electron picture is still a good distribution in many cases and in particular for metals, which is quite surprising since the Coulomb interaction between the electrons is a rather strong interaction. The reason for this will be discussed later on Fermi liquid theory.

At this step, we review the quasiparticle picture here (some details will be clarified in the following lectures). Many macroscopic systems, such as gases, liquids, and electrons in metals, particles behave as if they comprise nearly independent particles. Without interactions, we see the Green's function has a simple pole at $\omega = E_\lambda$. However, the system will have other states. If there is an interaction, we know from degenerated perturbation theory that these states will be coupled, the exact eigenstates being linear combinations of the original ones with energies spread out by the perturbation. Without the interactions, the single-particle state is stable (forever). Once the interaction turns on, all particles are coupled together, so original single-particle state is not the eigenstate of the system. First, there are many types of quasi-particles. We can discuss individual quasi-particle (e.g. electrons in normal metal) and collective quasipartlces (e.g. phonon). Second, in the many-body system, there could be many kinds of quasiparticles in a single system. Third, please note that, the statistics of quasi-particle could be the same as bare particle, but not always. (Think about cooper-pair in superconductivity, or density excitation

in Luttinger liquid, e.g.)

1.2.4 Connection with physical observables

There are several reasons for studying the Green's functions. First, the Feynman rules for finding the contribution of a n-th order perturbation theory are simpler for G than for other combinations of field operators (discussed in next section). Second, it is straight forward to study many observable properties.

By the help of the Green's function, we can calculate some physical observables easily. It seems nothing useful in the single particle physics, but it will be very helpful in many-particle physics especially for interacting systems, because we could get Green's function by perturbation method.

Considering a single-particle physical quantity

$$\hat{J} = \int dx J(x) = \int dx \psi_a^\dagger(x) J_{ab}(x) \psi_b(x). \tag{1.49}$$

One can think about example of density operator $\hat{n} = \sum_x n(x) = \sum_x \psi_b^\dagger(x)\psi_b(x)$.

Its averaged value (mean value) measured by the ground state

$$\langle 0|\hat{J}(x)|0\rangle = \sum_x J_{ab}(x)\langle 0|\psi_a^\dagger(x)\psi_b(x)|0\rangle \tag{1.50}$$

This can be written by the Green's function ($\zeta = -(+)$ is for fermion (boson))

$$\langle 0|\hat{J}(x)|0\rangle = \lim_{x\to x'} J_{ab}(x)\langle 0|\psi_a^\dagger(x',t)\psi_b(x,t)|0\rangle = \zeta \lim_{t'\to t^+}\lim_{x'\to x} \hat{J}_{ab}(x)\langle 0|T\psi_b(x,t)\psi_a^\dagger(x',t')|0\rangle \tag{1.51}$$

$$= \zeta i \lim_{t'\to t^+}\lim_{x'\to x} J_{ab}(x)G_{ba}(xt,x't') = \zeta i Tr[J(x)G(xt,xt^+)] \tag{1.52}$$

Here we use time-ordering operator to change the operator sequence in the above expression, so we have $t' > t$.

For example, for density operator and kinetic energy ($t^+ = t + 0^+$)

$$\langle n(x)\rangle = \zeta i G(xt, xt^+) \tag{1.53}$$

$$\langle T(x)\rangle = \zeta i \lim_{x'\to x}[\frac{-\nabla^2}{2m}G(xt, x', t^+)] \tag{1.54}$$

A slightly difficult example is the electron current. Recall the current operator is defined as

$$\mathbf{J}(\mathbf{r}) = \frac{ie}{2m}\sum_\lambda[(\nabla\psi^\dagger_\lambda(\mathbf{r}))\psi_\lambda(\mathbf{r}) - \psi^\dagger_\lambda(\mathbf{r}))(\nabla\psi_\lambda(\mathbf{r}))] \tag{1.55}$$

which can expressed as the Green's function also:

$$\begin{aligned}\langle\mathbf{J}(\mathbf{r})\rangle &= \frac{ie}{2m}\sum_\lambda \lim_{\mathbf{r}\to\mathbf{r}'}(\nabla_{\mathbf{r}'} - \nabla_{\mathbf{r}})\langle\psi^\dagger_\lambda(\mathbf{r}')\psi_\lambda(\mathbf{r})\rangle \\ &= \frac{ie}{2m}\sum_\lambda \lim_{t\to 0^-}\lim_{\mathbf{r}\to\mathbf{r}'}(\nabla_{\mathbf{r}'} - \nabla_{\mathbf{r}})(-i)G_\lambda(\mathbf{r}'t', \mathbf{r}0)\end{aligned} \tag{1.56}$$

At last, let us calculate the average kinetic energy density of non-interacting Fermi gas at zero temperature ($\epsilon_{\mathbf{k}} = E_{\mathbf{k}} - \mu$):

$$\begin{aligned}T &= \int dx\langle T(x)\rangle = -i\lim_{x'\to x}\frac{-\nabla^2}{2m}\left[\sum_{\mathbf{k}}\int\frac{d\omega}{2\pi}e^{-i\mathbf{k}\cdot(x-x')}e^{-i\omega t}G(\mathbf{k},\omega)\right]|_{t\to 0^-} \\ &= \int\frac{d^3k}{(2\pi)^3}\frac{k^2}{2m}[\int\frac{d\omega}{2\pi i}\frac{e^{-i\omega t}}{\omega - \epsilon_{\mathbf{k}} + i\eta sign(\epsilon_{\mathbf{k}})}]|_{t\to 0^-} \\ &= \int\frac{d^3k}{(2\pi)^3}\frac{k^2}{2m}\theta(k_F - k) \sim \frac{3}{5}E_F D(E_F)\end{aligned} \tag{1.57}$$

1.3 Diagram method for interacting Green's function

The preceding section defined the single-particle Green'sfunction and exhibited its relation to observable properties. So far, we have not discussed how to solve any fundamental many-body problem. We now consider how to calculate Green's function from the perturbation theory, in the interacting picture. We have seen that the Green's function, defined in the Heisenberg representation, can be converted to the interaction representation, which will be reviewed below. (See Eq. 1.68 and discussion there: green's function defined in the Heisenberg picture is equivalent to operators averaged over non-interacting ground state in the interacting picture) By expanding

the Green's function order-by-order, Wick's theorem is helpful to calculate the averages over the multiply operators. Then, these perturbative expansion is quite easy to formulated in the famous Feynman diagrams.

1.3.1 Perturbation expansion of Green's functions in the interacting picture

As we have mentioned, once we have Green's function, it is equivalent to solve the whole problem. However, in many cases (in interacting systems), exact solution of Green's function is not available. One possible way to solve the problem is some approximate method, e.g. perturbation method. Here we discuss the expansion on the base of perturbation calculations.

Let us recall the Schrodinger equation $i\partial_t\psi(t) = H\psi(t)$ again, and discuss the three different pictures.

Schrodinger picture: The wave function is time dependent $\psi_S(t) = e^{-iHt}\psi(0) = U(t)\psi(0)$ while the operator is time independent.

Heisenberg picture: The wave function is time independent while the operator is time dependent: $O_H(t) = e^{iHt}O_S e^{-iHt}$ and $i\partial_t O_H(t) = [O_H(t), H]$.

Interacting representation: Both wave function and operator are time dependent. The Hamiltonian is separated into two parts, $H = H_0 + V$, where H_0 is the unperturbed part and V is the interaction.

The operators and the wave function have a time dependence (And note that $\psi_I(0) = \psi_H(0) = \psi_S(0)$.)

$$O_I(t) = e^{iH_0t}O_S e^{-iH_0t} = e^{iH_0t}e^{-iHt}Q_H(t)e^{iHt}e^{-iH_0t} \equiv U(t)Q_H(t)U^+(t), \tag{1.58}$$

$$\psi_I(t) = e^{iH_0t}e^{-iHt}\psi_S(0) = e^{iH_0t}\psi_S(t) \equiv U(t)\psi_S(0) \tag{1.59}$$

Please note that $[H_0, V] \neq 0$ (if it is zero, the solution is trivial). Here we also define an operator $U(t) = e^{iH_0t}e^{-iHt}$, where $U(0) = 1$ for $t = 0$. One can check that, this definition ensure the physical observable doesnot depend on the choice of a specific picture: $\langle\psi_I^+(t)O_I(t)\psi_I(t)\rangle = \langle\psi_S^+(0)e^{iHt}O_S(0)e^{-iHt}\psi_S(0)\rangle$. Next show that the time dependence of the wave function is gov-

erned by the interactions

$$\begin{aligned}\partial_t\psi_I(t) &= ie^{iH_0t}(H_0 - H)e^{-iHt}\psi_S(0) = -ie^{iH_0t}Ve^{-iHt}\psi_S(0) \\ &= -ie^{iH_0t}Ve^{-iH_0t}[e^{iH_0t}e^{-iHt}\psi_S(0)] = -iV_I(t)\psi_I(t)\end{aligned} \tag{1.60}$$

This result proves the assertion that the time dependence of $\psi_I(t)$ is determined by $V_I(t)$. It follows the relation

$$\partial_t U(t) = ie^{iH_0t}(H_0 - H)e^{-iHt} = -ie^{iH_0t}Ve^{-iH_0t}e^{iH_0t}e^{-iHt} = -iV_I(t)U(t) \tag{1.61}$$

In order to solve this equation, one way of proceeding is by integrating both sides of the equation with respect to time (from here on, we neglect the symbol "I"):

$$\begin{aligned}U(t) - U(0) &= -i\int_0^t dt_1 V(t_1)U(t_1) \to U(t) = 1 - i\int_0^t dt_1 V(t_1)U(t_1) \\ \Rightarrow U(t) &= 1 - i\int_0^t dt_1 V(t_1) + (-i)^2\int_0^t dt_1\int_0^{t_1} dt_2 V(t_1)V(t_2) + ... \\ &= \sum_{n=0}^{\infty}(-i)^n\int_0^t dt_1\int_0^{t_1} dt_2 ...\int_0^{t_{n-1}} V(t_1)V(t_2)..V(t_n)\end{aligned} \tag{1.62}$$

At this point it is convenient to introduce the time-ordering operator T (see Eq. 1.22), which should not be confused with the temperature. T operator acts upon a group of time-dependent operators, $T[V(t_1)V(t_2)V(t_3)] = V(t_3)V(t_1)V(t_2), t_3 > t_1 > t_2$.

Next we can prove this relation

$$\begin{aligned}\frac{1}{2!}\int_0^t dt_1\int_0^t dt_2 T[V(t_1)V(t_2)] &= \frac{1}{2!}\int_0^t dt_1\int_0^{t_1} dt_2 V(t_1)V(t_2) + \frac{1}{2!}\int_0^t dt_2\int_0^{t_2} dt_1 V(t_2)V(t_1) \\ &= \int_0^t dt_1\int_0^{t_1} dt_2 V(t_1)V(t_2) \\ \frac{1}{3!}\int_0^t dt_1\int_0^t dt_2\int_0^t dt_3 T[V(t_1)V(t_2)V(t_3)] &= \int_0^t dt_1\int_0^{t_1} dt_2\int_0^{t_2} dt_3 V(t_1)V(t_2)V(t_3)\end{aligned} \tag{1.63}$$

Thus we can write the time evolution operator as

$$\begin{aligned} U(t) &= \sum_{n=0}^{\infty}(-i)^n \int_0^t dt_1 \int_0^{t_1} dt_2 ... \int_0^{t_{n-1}} V(t_1)V(t_2)..V(t_n) \\ &= 1 + \sum_{n=1}^{\infty}(-i)^n \frac{1}{n!} \int_0^t dt_1 \int_0^t dt_2 ... \int_0^t T[V(t_1)V(t_2)..V(t_n)] \\ &= T\exp[-i\int_0^t dt_1 V(t_1)] \end{aligned} \tag{1.64}$$

At this moment, there is a problem in that the formalism is based on the wave function $\psi(0) \equiv |0\rangle$ which is not yet known (due to interaction V). Now in the interaction representation set $H = H_0 + V$, where H_0 is chosen to be sufficiently simple that its eigenvalues and eigenstates are known. Let the lowest eigenvalue of H_0-its ground state-be denoted $|0\rangle_0$. Somehow the unknown wave function $|0\rangle$ must be determined in terms of the known wave function $|0\rangle_0$ (Or, let us assume $H = H_0$ when $t \to -\infty$ and the interaction V turns on very slowly until $t = 0$: $H = H_0 + e^{\epsilon t}V$). The relationship between the two ground states $|0\rangle$ and $|0\rangle_0$ at zero temperature was established by Gell-Mann and Low (1951):

$$\psi(0) = U(0, -\infty)\phi_0(t = -\infty), \quad or |0\rangle = U(0, -\infty)|0\rangle_0 \tag{1.65}$$

$$\psi(t) = U(t, 0)\psi(0), \tag{1.66}$$

For $t > 0$, we assume the interaction slowly turns down, and in the limit of $t \to -\infty$, interaction disappears. The whole process is adiabatic, and the total energy is conserved, in the limit of $t \to \infty$, the system is still in the ground state, up to a phase factor: $\psi(\infty) = U(\infty, -\infty)\phi_0(-\infty) = e^{-iL}\psi_0(-\infty)$ or $e^{-iL} =_0 \langle 0|U(\infty, -\infty)|0\rangle_0$.

Now we calculate the averaged value of a Heisenberg operator:

$$\begin{aligned} &\langle A(t)\rangle = \langle \psi_H|A_H(t)|\psi_H\rangle = \langle \psi_I(t)|A_I(t)|\psi_I(t)\rangle \\ &= \langle \infty|U(\infty, 0)U(0, t)A_I(t)U(t, 0)U(0, -\infty)| - \infty\rangle \\ &= e^{iL}\,{}_0\langle 0|U(\infty, 0)U(0, t)A_I(t)U(t, 0)U(0, -\infty)|0\rangle_0 \\ &= \frac{{}_0\langle 0|U(\infty, t)A_I(t)U(t, -\infty)\rangle_0}{{}_0\langle 0|U(\infty, -\infty)|0\rangle_0} = \frac{{}_0\langle 0|T[A_I(t)U(\infty, -\infty)]\rangle_0}{{}_0\langle 0|U(\infty, -\infty)|0\rangle_0} \end{aligned} \tag{1.67}$$

Here the time-ordered operator is used again to simplify the expression.

For time-sequence of two operators, we have a similar form

$$\langle\psi_H|TA_H(t_1)B_H(t_2)|\psi_H\rangle = \frac{{}_0\langle 0|T[A_I(t_1)B_I(t_2)U(\infty,-\infty)]\rangle_0}{{}_0\langle 0|U(\infty,-\infty)|0\rangle_0} \tag{1.68}$$

As an example that will be frequently used below, let us explicitly write down the definition of Green's function at $t > t'$: By creating one particle at t' and then evolve it under H, and check the proprobility at time t:

$$\begin{aligned}
&\langle_0|\psi_{Ha}(xt)\psi^\dagger_{Hb}(x't')|_0\rangle = \\
&= \frac{{}_0\langle 0|U(\infty,t)[U(t,0)\psi_{Ha}(xt)U(0,t)]U^+(0,t)U^+(t',0)[U(t',0)\psi^\dagger_{Hb}(x't')U(0,t')]U(t',-\infty)|0\rangle_0}{{}_0\langle 0|U(\infty,-\infty)|0\rangle_0} \\
&= \frac{\langle\Psi_I(t)|\psi_{Ia}(xt)U(t,t')\psi^\dagger_{Ib}(x't')|\Psi_I(t')\rangle}{{}_0\langle 0|U(\infty,-\infty)|0\rangle_0}
\end{aligned} \tag{1.69}$$

Similarly, if $t < t'$, the field operator creates a hole at t, and the system propagates according to the full hamiltonian. These holes can be interpreted as particles going backward in time.

At last, let us stress why we should make efforts to get relationship in Eq. 1.68. In the original definition, the Green's function is to take average over $|0\rangle$, which is the ground state of $H = H_0 + V$. The key issue is, usually we cannot solve exactly H and $|0\rangle$. (Once we solve it, it recovers the "single-particle" physics as discussed in the previous section). Instead, we can often solve a simplified version of the many-body Hamiltonian H_0 where the ground state $|0\rangle_0$ is easy to know. Intuitively, we can "adiabatically" evolve (unknown) $|0\rangle$ from a known $|0\rangle_0$, by setting $H = H_0 + e^{\epsilon t}V$. In this context, the Gell-MannLow theorem enables us to relate the Greens function of the interacting system to the Greens function of the non-interacting system at $t = -\infty$. (To reach it, we have to bear two costs: One is we have to use interaction picture; the other one is we relate the Greens function a set of operators multiplied by the U-matrix.) These are the base of Feynman diagram below.

1.3.2 Wick's theorem

As we have seen, in the interaction picture, the Green's function can be expressed as a perturbative expansion. Explicitly, the Green's function is evaluated by expanding $U(\infty, -\infty)$:

$$\begin{aligned} iG(\lambda, t-t') &= \frac{{}_0\langle|T[\psi_\lambda(t)\psi_\lambda^\dagger(t')U(\infty,-\infty)]\rangle_0}{{}_0\langle 0|U(\infty,-\infty)|0\rangle_0} \\ &= \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_{-\infty}^{\infty}dt_1...\int_{-\infty}^{\infty}dt_n\frac{{}_0\langle|T\psi_\lambda(t)V(t_1)V(t_2)..V(t_n)\psi_\lambda^\dagger(t')|\rangle_0}{{}_0\langle|U(\infty,-\infty)|\rangle_0} \end{aligned} \tag{1.70}$$

$V(t)$ is the operator in interacting picture. Here we have used the time-evolution operator Eq. 1.62. (Here and henceforth, the subscript I will be omitted, since we shall consistently work in the interaction picture.) Let us, for the moment, ignore the phase factor ${}_0\langle|U(\infty,-\infty)|\rangle_0$. It will be taken care of in the next few section. The immediate aim is to learn how to evaluate time-ordered brackets such as ${}_0\langle|T\psi_\lambda(t)V(t_1)V(t_2)..V(t_n)\psi^\dagger(t')|\rangle_0$.

Suppose that $V(t)$ is some kind of electron-electron interaction, such as $V(t) \sim \sum_{k',k,q} V(q)\psi^\dagger_{k+q}\psi^\dagger_{k'-q}\psi_{k'}\psi$ In this case the time-ordered bracket contains many creation operators and many destruction operators. It is a very difficult task to evaluate this bracket: there are many possible time orderings and many possible pairings between creation and destruction operators. First note that these brackets always contain the same number of creation and destruction operators. One is always trying to evaluate the product of creation operators and destruction operators between the ground state, like

$${}_0\langle|T\psi_m(t)\psi_1^\dagger(t_1)...\psi_n(t_n)\psi_{m'}^\dagger(t')|\rangle_0 \tag{1.71}$$

The effect of a creation operator $\psi^\dagger_{m'}(t')$ is to put an electron into the state m'. The system must be back in the ground state before the final operator of ${}_0\langle|$, so that one of the destruction operators $\psi_m(t)$ must destroy the state m' and $m = m'$ for some m. For example, ${}_0\langle|T\psi_m(t)\psi^\dagger_{m'}(t')|\rangle_0$ equals zero unless $m = m'$. ${}_0\langle|T\psi_m(t)\psi^\dagger_{m_1}(t_1)\psi_{m_2}(t_2)\psi^\dagger_{m'}(t')|\rangle_0$ equals zero unless $m = m_1, m_2 = m'$ or $m = m', m_1 = m_2$. Therefore, we see in the above equation, only a limited number of these combinations are physically interesting. Our aim is to sort these in a simple way to identify the important terms. This sorting is achieved with the help of some theorems which simplify the procedures. The first of these is Wick's theorem.

Let us present the statement from the Wick's theorem, and give a formal proof at the end.

This theorem is really just an observation that the time ordering can be taken care of in a simple way. Wick's theorem states that, in making all the possible pairings between creation and destruction operators, each pairing should be time-ordered. The time ordering of each pair gives the proper time ordering to the entire result. For example,

$$\begin{aligned}
&\langle|T\psi_m(t)\psi^\dagger_{m_1}(t_1)\psi_{m_2}(t_2)\psi^\dagger_{m'}(t')|\rangle \\
&= \langle|T\psi_m(t)\psi^\dagger_{m_1}(t_1)|\rangle\langle|T\psi_{m_2}(t_2)\psi^\dagger_{m'}(t')|\rangle - \langle|T\psi_m(t)\psi^\dagger_{m'}(t')|\rangle\langle|T\psi_{m_2}(t_2)\psi^\dagger_{m_1}(t_1)|\rangle \\
&= \delta_{m,m_1}\delta_{m_2,m'}\langle|T\psi_m(t)\psi^\dagger_m(t_1)|\rangle\langle|T\psi_{m'}(t_2)\psi^\dagger_{m'}(t')|\rangle - \delta_{m,m'}\delta_{m_1,m_2}\langle|T\psi_m(t)\psi^\dagger_m(t')|\rangle\langle|T\psi_{m_2}(t_2)\psi^\dagger_{m_1}(t_1)|\rangle
\end{aligned} \tag{1.72}$$

Note that there is a time-ordering operator T in each of the two pairing brackets. For n operators of each kind there are $n!$ possible pairings. Also note that within a pairing bracket, the labels m, m' etc., must be the same. These labels denote eigenstates, so the creation and destruction operators must refer to the same state.

A few simple rules should be kept in mind when making these pairings. The first is that a sign change occurs each time the positions of two neighboring Fermi operators are interchanged. One keeps count of the number of interchanges needed to achieve the desired pairing. An odd number of interchanges is the origin of the minus sign in the second term of the example above.

The second rule concerns the time ordering of combinations of operators representing different excitations. For example, consider the following mixture of phonon and electron operators: $\langle|T\psi_n(t)\psi^+_q(t_1)A_x(t_1)\psi_m(t_2)\psi^+_p(t_3)A_y(t_2)|\rangle = \langle|T\psi_n(t)\psi^+_q(t_1)\psi_m(t_2)\psi^+_p(t_3)|\rangle\langle TA_x(t_1)A_y(t_2)\rangle$. Because electron operators commute with phonon operators, it is not important how they are ordered with respect to each other.

The third rule is a method of treating the "time ordering" of two operators which occur at the same time, such as

$$\langle T\psi^\dagger_n(t_1)\psi_m(t_1)\rangle = \delta_{n=m}\langle\psi^\dagger_n(t_1)\psi_m(t_1)\rangle = \delta_{m=n}n_F(\xi_m) \tag{1.73}$$

and the term is just the number operator which is independent of time. This convention is dependent on the convention used to write down the Hamiltonian.

When two electron operators have different time arguments in a pairing, it is conventional

to put the creation operator to the right:

$$\langle T\psi_n(t_2)\psi_m^\dagger(t_1)\rangle = iG^{(0)}(\lambda_m, t_2 - t_1) \tag{1.74}$$

All of the pairing brackets for electron operators are either Green's functions or else number operators. The previous examples can also be written in terms of Green's functions:

$$\begin{aligned}&\langle|T\psi_m(t)\psi_{m_1}^\dagger(t_1)\psi_{m_2}(t_2)\psi_{m'}^\dagger(t')|\rangle \\ &= \delta_{m,m_1}\delta_{m_2,m'}iG^{(0)}(m, t-t_1)iG^{(0)}(m_1, t_2 - t') - \delta_{m,m'}\delta_{m_1,m_2}iG^{(0)}(m, t-t')iG^{(0)}(m_1, t_2 - t_1)\end{aligned} \tag{1.75}$$

In summary, Wick's theorem states that a time-ordered bracket may be evaluated by expanding it into all possible pairings. Each of these pairings will be a time-ordered Green's function or a number operator n_F or n_B. This expansion gets the correct time-ordering for the entire brackets. Wick's theorem is valid only when the Hamiltonian H_0 is bilinear in creation and destruction operators.

We will present a prove for Wick's theorem in the zero temperature case, as shown below. First, we define two operator sequences: T-sequence: the operator with latest time is on the left side that we have clarified before. N-sequence: creation operator on the left side

$$\begin{aligned}: \psi_1^\dagger\psi_2 :&= \psi_1^\dagger\psi_2 \\ : \psi_1\psi_2^\dagger :&= (-)^P\psi_2^\dagger\psi_1\end{aligned} \tag{1.76}$$

where P is number exchange of fermionic operators.

The idea of Wick's theorem is, to reorder the sequence of operators, and let the creation operators on the left and the decreation operator on the right. When two different operators exchange their sequence, there is an additional term term coming from the commutation relation. Wick's theorem contains all terms when reorder time-sequence operator to normal-sequence operator. A normal-ordered product of field operators is especially convenient because its expectation value in the unperturbed ground state vanishes identically.

Second, we define the difference between T-sequence and N-sequence is Contractions:

$$T(XY) =: XY : + \overbrace{XY}. \tag{1.77}$$

Thus we have some properties: 1) For operator X,Z, if Z is decreation operator and $t_x > t_z$, the contraction should be zero: $\overbrace{XZ} = T[XZ] - : XZ := XZ - XZ = 0$. 2) Any contraction between two operators is a c-number, either zero or Green's function (to be shown in next section).

To present the Wick's theorem, we first prove a lemma: There is a N-sequence operators and a operator Z labeled with a time earlier than the times for others, $(t_z < t_x, t_y, ..)$, then

$$: UV...XY : Z =: UV...XYZ : + : \overbrace{UV...XYZ} : + : U\overbrace{V...XYZ} : + ... + : UV...X\overbrace{YZ} : \tag{1.78}$$

(a)When Z is destruction operator, the prove is straight forward, since contraction $\overbrace{XZ} = 0$. (b) Next we only need discuss the case of Z as creation operator. In this case, we can first prove the case of Z as creation operator and $XY...UV$ are all destruction operators. (If there are other creation operators, we just multiply the creation operator on the very left side, and consider the contraction between two creation operators is identically zero.) In this case, we use the induction method: we first assume that the lemma equation is true for n operators and prove it for $n+1$ operators. (When $n = 1$, it is true: $: Y : Z = T[YZ] = \overbrace{YZ} + : YZ :$.) Multiply the lemma equation on the left by another destruction operator D having a time *later* than that of Z ($t_D > t_Z$):

$$\begin{aligned}
&D : UV..XY : Z \\
&= D\left[: UV..X\overbrace{YZ} : + : UV..\overbrace{XYZ} : + ... + : \overbrace{UV...XYZ} : + : UV...XYZ :\right] \\
&=: DUV..X\overbrace{YZ} : + : DUV..\overbrace{XYZ} : + ... + : D\overbrace{UV...XYZ} : + D : UV...XYZ : \\
&\overset{!}{=}: DUV..X\overbrace{YZ} : + : DUV..\overbrace{XYZ} : + ... + : D\overbrace{UV...XYZ} : + : \overbrace{DUV..XYZ} : + : DUV...XYZ : \\
&=: DUV..XY : Z
\end{aligned} \tag{1.79}$$

In the second line, we just move D in the normal ordering, because D is also destroy operator,

same as $U, V, ..., X, Y$, but not Z. In the fourth line, we used the relation:

$$\begin{aligned} D:UV...XYZ &:= (-)^P DZUV...XY = (-)^P T[DZ]UV...XY \\ &= (-)^P \overbrace{DZ}UV...XY + (-)^{P+x} : ZD : UV...XY \\ &= (-)^{2P} \overbrace{DUV...XYZ} + (-)^{2(P+x)} : DUV...XYZ : \\ &=: \overbrace{DUV...XYZ} : + : DUV...XYZ : \end{aligned} \tag{1.80}$$

where in the first line above, we used Z is moved to the left within the normal-ordered product, introducing a signature factor $(-1)^P$. The sequence is correct, so normal-ordered symbol can be dropped. Furthermore, the product DZ is already time ordered by assumption. The second line follows from the definition of a contraction, with a factor $(-1)^x$ arising from the interchange of D and Z. The term in the third line is in normal order, because UV...XY are all destruction operators.

Thus the lemma equation is valid for $n+1$ operators, i.e. the lemma equation is valid generally.

With these preparation, we are ready to prove Wick's theorem now:

$$\begin{aligned} T[XYZ...UVW] =: XYZ...UVW : + : \overbrace{XY}Z...UVW : + ... + : \overbrace{XYZ...UVW} : + : XYZ...\overbrace{UVW} : \\ + : \overbrace{XY}Z...\overbrace{UVW} : + : \overbrace{XY}\overbrace{Z...UVW} : + ... \\ =: XYZ...UVW : + (allcontractions) \end{aligned} \tag{1.81}$$

That is, the time-ordering operators is equal to normal-ordering and normal-ordering with all kinds of contractions. "All kinds of contractions" means contraction between two operators, two pairs of contractions, and more pairs.

We again use the induction method. It is obviously true for two operators:

$$T[UV] =: UV : + \overbrace{UV} \tag{1.82}$$

Assuming it is true for n operators, we multiply it on the right by an operator Ω with a time

earlier than the others ($t_\Omega < t_{U,V,...}$):

$$\begin{aligned} &T[UVW...XYZ]\Omega = \\ &= \left[: UVW...XYZ : + : \overbrace{UV}W...XYZ : + ... : allcontractions : \right] \Omega \\ &=: UVW...XYZ\Omega : + : allcontractions := T[UVW...XYZ\Omega] \end{aligned} \tag{1.83}$$

In the second line, we use Wick's theorem for n operators. In the third line, we use the lemma to put Ω in the normal ordering, and collect all other contractions. One will see the last line is the Wick's theorem for n+1 operators. The restriction on the time of the operator Ω can now be removed by simultaneously reordering the operators in each term. Again the sign conventions give the same overall sign on both sides of the equation, which therefore remains correct.

In the above, we see Wick theorem is for changing time-ordering operators to normal ordering operators. (Actually, Wick's theorem is for a theorem for operators.) If we further evaluate the operators in the ground state, all terms vanish except for one contraction term:

$$\begin{aligned} &\langle 0|T[ABCD...XYZ]|0\rangle = \\ &\overbrace{AB}\overbrace{CD} + ... + \overbrace{A\overbrace{BC}D}... + ... \end{aligned} \tag{1.84}$$

Thus, the averaged value is multiply of contractions. Therefore, arbitary T-ordering operators in ground state average is sum up of multiply of contractions (physical meaning of contraction will be clarified below).

1.3.3 Feynmann Diagram

Wick's theorem allows us to evaluate the exact Green's function as a perturbation expansion involving only contracted field operators in the interaction picture. These contractions are just the free Green's functions (explain below).

Let us consider the contraction, or pairing, of two operators. If both operators here are of the same sort (both creation or both annihilation), the contraction is identically zero. Indeed,

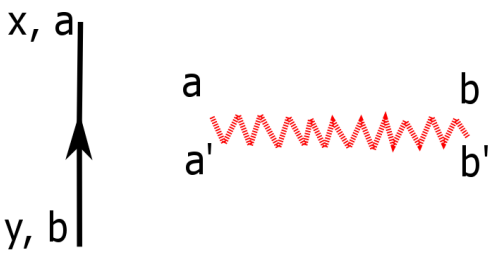


Figure 1.2: Feynman diagrams in the real space: Solid line denotes the Green's function heading from y to x. Wave line is the two-body interaction.

then the normal ordering does not affect their product, and

$$\begin{aligned}\overbrace{\psi_1\psi_2} &= T\psi_1\psi_2 - :\psi_1\psi_2: \\ &= \theta(t_1-t_2)\psi_1\psi_2 \mp \theta(t_2-t_1)\psi_2\psi_1 - \psi_1\psi_2 = [\theta(t_1-t_2)+\theta(t_2-t_1)]\psi_1\psi_2 - \psi_1\psi_2 = 0\end{aligned} \tag{1.85}$$

On the other hand, a contraction of conjugate field operators is a number: taking into account that the operators are in interaction representation and their time dependence is trivial (see Eq. 1.58), we see that, for example,

$$\begin{aligned}\psi_1^\dagger\overbrace{(t_1)\psi_2(t_2)} &= \left[T[\psi_1^\dagger(t_1)\psi_2(t_2)] - :\psi_1^\dagger(t_1)\psi_2(t_2):\right] \\ &= \left[\theta(t_1-t_2)\psi_1^\dagger(t_1)\psi_2(t_2) - \theta(t_2-t_1)\psi_2(t_2)\psi_1^\dagger(t_1) - \psi_1^\dagger(t_1)\psi_2(t_2)\right] \\ &= \left[\theta(t_2-t_1)+\theta(t_1-t_2)\right]\psi_1^\dagger(t_1)\psi_2(t_2) - \theta(t_2-t_1)\{\psi_2(t_2),\psi_1^\dagger(t_1)\} - \psi_1^\dagger(t_1)\psi_2(t_2) \\ &= -\theta(t_2-t_1)\{\psi_2(t_2),\psi_1^\dagger(t_1)\} \\ &= -\sum_{k,q} e^{iE_kt_2}e^{-iE_qt_1}\theta(t_2-t_1)\{\psi_k,\psi_q^\dagger\} \\ &= -\sum_{k,q} e^{iE_kt_2}e^{-iE_qt_1}\theta(t_2-t_1)\delta_{kq} = -\sum_k e^{iE_k(t_2-t_1)}\theta(t_2-t_1)\end{aligned} \tag{1.86}$$

and all the operator terms cancel, so it is a c-number. This is an important fact, that the contraction of Fermi/Bose field operators is a usual number, and this number is nothing but free Green's function:

$$\begin{aligned}\overbrace{\psi_1^\dagger\psi_2} &= \langle 0|\overbrace{\psi_1^\dagger\psi_2}|0\rangle = \langle 0|T\psi_1^\dagger\psi_2 - :\psi_1^\dagger\psi_2:|0\rangle = \langle 0|T\psi_1^\dagger\psi_2|0\rangle = -iG_0(12) \\ \overbrace{\psi_2\psi_1^\dagger} &= \langle 0|\overbrace{\psi_2\psi_1^\dagger}|0\rangle = \langle 0|T\psi_2\psi_1^\dagger - :\psi_2\psi_1^\dagger:|0\rangle = \langle 0|T\psi_2\psi_1^\dagger|0\rangle = iG_0(21)\end{aligned} \tag{1.87}$$

Now we have all preparation for evaluate full Green's functions. But, even with Wick's

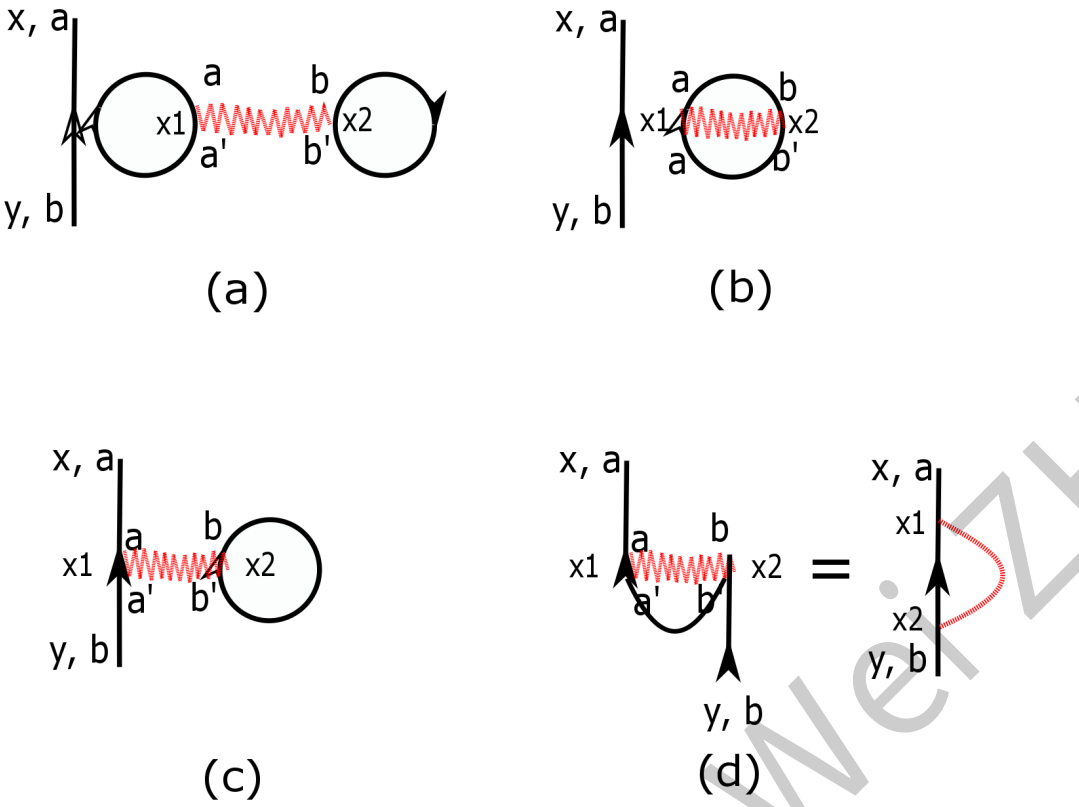


Figure 1.3: First-order Feynman diagrams in the real space.

theorem, perturbation expansion involving only contracted field operators (see Eq. 1.70) is a huge task (see example below). Feynman introduced the idea of representing the kind of terms in Wick's theorem by drawings. These drawings, called diagrams, are extremely useful for providing an insight into the physical process which these terms represent. These diagrams can be drawn both for the Green's function depending on time $G(t)$ as well as for functions which are Fourier transformed and depend on frequency $G(\omega)$.

This expansion can be analyzed directly in coordinate space, or(for a uniform system) in momentum space. We will discuss them separately.

Feymann diagram in spatial space

Let us consider Eq. 1.70, and assume the interaction as two-body case

$$V(t) = \frac{1}{2} \sum_{\lambda,\lambda',\mu,\mu'} \int d^4x d^4y \psi^\dagger_\lambda(x) \psi^\dagger_\mu(y) V(x-y)_{\lambda\lambda'\mu\mu'} \psi_{\mu'}(y) \psi_{\lambda'}(x) \tag{1.88}$$

We see there are four operators in each $V(t)$, so n-th perturbative term contains 2n+1 pair of operators. For $n = 0$ case, we have

$$iG^{(0)}_{ab}(xt, yt') =_0 \langle 0|T\psi_a(xt)\psi^\dagger_b(yt')|0\rangle_0 \tag{1.89}$$

which is free green's function. For $n = 1$ case, we calculate the first order in Eq. 1.70,

$$\begin{aligned} iG^{(1)}_{ab}(x,y) &= \frac{-i}{<s>}\frac{1}{2}\int d^4x_1 d^4x_2 U(x_1,x_2)_{\lambda,\lambda',\mu,\mu'}\langle 0|T[\psi_a(x)\psi^\dagger_b(y)\psi^\dagger_\lambda(x_1)\psi^\dagger_\mu(x_2)\psi_{\mu'}(x_2)\psi_{\lambda'}(x_1)]|0\rangle \\ &= \frac{-i}{<s>}\frac{1}{2}\int d^4x_1 d^4x_2 U(x_1,x_2)_{\lambda,\lambda',\mu,\mu'}\times \\ &[iG^0_{ab}(x,y)iG^0_{\mu'\mu}(x_2,x_2)iG^0_{\lambda'\lambda}(x_1,x_1) - iG^0_{ab}(x,y)iG^0_{\mu'\lambda}(x_2,x_1)iG^0_{\lambda'\mu}(x_1,x_2) \\ &+ iG^0_{a\lambda}(x,x_1)iG^0_{\lambda'\mu}(x_1,x_2)iG^0_{\mu'\beta}(x_2,y) - iG^0_{a\lambda}(x,x_1)iG^0_{\lambda'b}(x_1,y)iG^0_{\mu'\mu}(x_2,x_2) \\ &+ iG^0_{a\mu}(x,x_2)iG^0_{\mu'\lambda}(x_2,x_1)iG^0_{\lambda'b}(x_1,y) - iG^0_{a\mu}(x,x_2)iG^0_{\mu'b}(x_2,y)iG^0_{\lambda'\lambda}(x_1,x_1)] \end{aligned} \tag{1.90}$$

There are six terms in total, using the Wick's theorem. For $n = 2$ case, there are $5! = 120$ terms. And in higher orders, the number grows very quickly, so we need some method to help us.

Now we introduce the diagram method, which helps greatly simply the discussion. We express each term in perturbation by a diagram. $G^{(0)}(x, y)$ is an arrowed line, where the arrow means from the second variable to the first variable. The potential is a waved line. Interaction $V(x_i - x_j)$ is represented by a dashed line connecting x_i and x_j. As shown in Fig. 1.2.

The Eq. 1.90 can be expressed graphically in Fig. 1.3 term by term. The representation of Feynman diagram has the following properties:

- In Fig. 1.3, a,b,c contain a Green's function at the same time, which is indicated by a solid line closed on itself. Such a term, however,arises from a contraction of two fields within the interaction hamiltonian $V(t)$. In consequence,the Green's function at equal times must be interpreted as

$$iG^{(0)}_{ab}(x,x) = \lim_{t'\to t^+}\langle 0|T[\psi_a(x,t)\psi^\dagger_b(xt')]|0\rangle = -\langle 0|\psi^\dagger_b(xt')\psi_a(x,t)|0\rangle = -\delta_{ab}n^{(0)}(x) \tag{1.91}$$

  where $n(x)$ is the particle density in the unperturbed ground state.

- The Feynman diagram can be divided into two classes: connected and disconnected diagrams. For example, Fig. 1.3 a, b are disconnected diagrams, containing subunits that are

not connected to the rest of the diagram by any lines. The connected diagram contains $\psi_a(x), \psi_b(y)$ connects with $V(t_1)$ and $V(t_1)$ connects with $V(t_2)$, and so on. Disconnected graph means at least one opertors in V is not connected with $\psi_a(x), \psi_b(y)$. Now we consider to bipartition the green's function as

$$
\begin{aligned}
&-i\frac{(-i)^n}{n!}\frac{n!}{m!(n-m)!}\int dt_1..dt_m\langle 0|T[\psi_a(x)\psi_b^+(y)V(t_1)...V(t_m)]|0\rangle_c \int dt_{m+1}..dt_n\langle 0|T[V(t_{m+1})...V(t \\
&= -i\frac{(-i)^m}{m!}\int dt_1..dt_m\langle 0|T[\psi_a(x)\psi_b^+(y)V(t_1)...V(t_m)]|0\rangle_c\times \\
&\quad \frac{(-i)^{n-m}}{(n-m)!}\int dt_{m+1}..dt_n\langle 0|T[V(t_{m+1})...V(t_n)]|0\rangle_{dc} \\
&= -i\frac{(-i)^m}{m!}\int dt_1..dt_m\langle 0|T[\psi_a(x)\psi_b^+(y)V(t_1)...V(t_m)]|0\rangle_c\times \\
&\quad \{1-i\int_{-\infty}^{\infty}dt_{m+1}\langle 0|TV(t_{m+1})|0\rangle - \frac{1}{2}\int_{-\infty}^{\infty}dt_{m+1}dt_{m+2}\langle 0|TV(t_{m+1})V(t_{m+2})|0\rangle + ...\}_{dc} \\
&\Rightarrow \langle 0|T[\psi_a(x)\psi_b^+(y)U(\infty,-\infty)]|0\rangle_c\langle 0|U(\infty,-\infty)|0\rangle
\end{aligned}
\tag{1.92}
$$

where "connect" means connected diagram (with external line) and "disconnect" means disconnected part. This is equivalent to bipartition n operators into m and n-m part. Amazingly, the disconnected part (by expansion) is the same as the $\langle 0|U(\infty,-\infty)|0\rangle$. And this part will be canceled by $\langle 0|U(\infty,-\infty)|0\rangle$ in the denominator. If only consider the connected diagrams, the denominator of $G_{ab}(x,y)$ can be omitted. These are the celebrated Feynman diagrams, and we shall now derive the precise rules that relate the diagrams to the terms of the perturbation series. The diagram explanation can be seen in Fig. 1.5.

- For n-th order diagram, there are n interaction lines. Permutation of the position of these n interaction lines leads to a factor of $n!$ (actually they are topological equivalent). This will cancel the factor $\frac{1}{n!}$ (see Fig. 1.4). This is only correct for connected diagrams.
- For each interaction line, by exchanging x and x' doesnot change the results. They differ only in that x and x'(and the corresponding matrix indices) are interchanged, whereas the potential is symmetric under this substitution. It is therefore sufficient to retain just one diagram of each type, simultaneously omitting the factor 1/2 in front of Eq. 1.90 which reflects the factor 1/2 in the interaction potential.

To sum up, We therefore obtain the rules for n-th order contribution to the Green's function $G_{ab}(x,y)$:

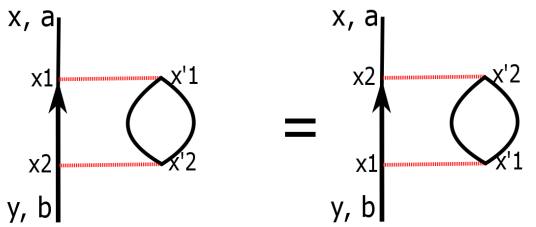


Figure 1.4: Feynman diagrams in the real space. Exchange the interaction lines lead to equivalent diagrams.

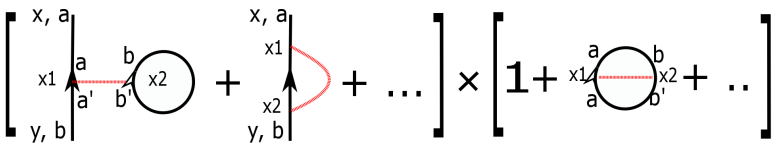


Figure 1.5: Feynman diagrams of disconnected diagrams in the real space.

1. Draw all topologically distinct connected diagrams with n interaction lines V (with $2n$ vertices; interaction line is equal-time), and $2n+1$ directed Green's functions $G^{(0)}$;
2. Label each vertex with a four-dimensional space-time point $x_i = (\mathbf{x}_i, t_i)$;
3. Each solid line represents a Green'sfunction running from y to x;
4. Each wave line represents an interaction $V_{\lambda\lambda',\mu\mu'}\delta(t_x - t_y)$;
5. Integrate all internal variables over space and time: $\int d^d\mathbf{x}_i \int dt_i$;
6. There is a spin matrix product along each continuous fermion line,including the potentials at each vertex;
7. Affix a sign factor $(-1)^F$ to each term, where F is the number of closed fermion loops in the diagram (this sign comes from the exchange ordering of operators according to the Wick's theorem);
8. To compute $G(x,y)$ as sign a factor $(-i)(-i/\hbar)^n(i)^{2n+1} = (i/\hbar)^n$ to each n-th order term. (index $2n+1$ comes from $2n+1$ number of Green's functions)

Finally, we know the first order Feynman diagram is shown in Fig. 1.6:

$$G^{(1)}_{ab}(x,y) = i\int dx_1 dx_2\{(-1)G^{(0)}_{a\lambda}(x,x_1)V_{\lambda\lambda',\mu\mu'}(x_1,x_2)G^{(0)}_{\lambda' b}(x_1,y)G^{(0)}_{\mu\mu'}(x_2,x_2)+$$
$$G^{(0)}_{a\lambda}(x,x_1)V_{\lambda\lambda',\mu\mu'}(x_1,x_2)G^{(0)}_{\lambda'\mu}(x_1,x_2)G^{(0)}_{\lambda' b}(x_2,y)\} \tag{1.93}$$

Similarly, if we charge the interaction term to different forms, such as impurity potential $V = \int dx V_{ab}(x)\psi_a^\dagger(x)\psi_b(x)$, or electron-phonon interaction $V = \sum_a \gamma \int dx \psi_a^\dagger(x)\psi_a(x)\varphi(x)$, we will generate a new form of Feymann diagrams. We will study such kinds of problem in the latter chapters.

Feynman diagrams in the momentum space

For translational invariant systems, it is convenitent to work in the momentum space. Here clarify the rules in momentum space. We make a Fourier transformation on zero-th order Green's function,

$$G_{ab}(x,y) = (2\pi)^{-D} \int d^D k e^{ik(x-y)} G_{ab}(k), \tag{1.94}$$

where we used the notion $d^D k = d^d k d\omega, k \cdot x = \mathbf{k}\cdot\mathbf{x} - \omega\mathbf{t}$.

We assume the interaction has the form $U(x,x') = V(x-x')\delta(t-t')$. We make the Fourier transformation as

$$U(x,x')_{aa',bb'} = (2\pi)^{-D} \int d^D k e^{ik\cdot(x-x')} U(k)_{aa',bb'} \tag{1.95}$$

$$U(k)_{aa',bb'} = \int d^d x e^{-ik\cdot x} V(x)_{aa',bb'} \tag{1.96}$$

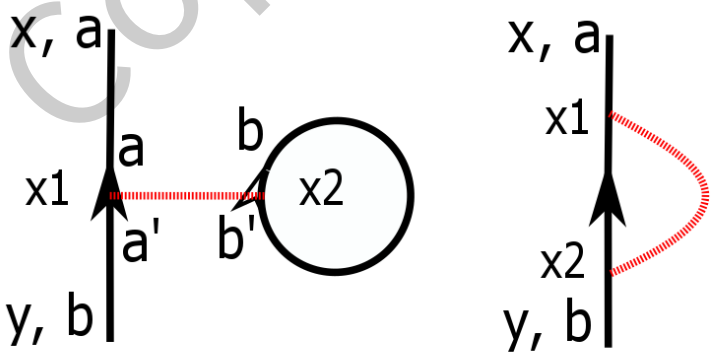


Figure 1.6: Connected Feynman diagrams of first-order corrections in the real space.

Next we take the example of first order Feynman diagram as shown in Fig. 1.6(b):

$$
\begin{aligned}
&G_{ab}(x,y) = i\int d^4x_1 d^4x_1'(2\pi)^{-16}\int d^4k d^4p d^4p_1 d^4q\times \\
&G^0_{a\lambda}(k)U(q)_{\lambda\lambda',\mu\mu'}G^0_{\lambda'\mu}(p)G^0_{\mu'\beta}(p_1)e^{ik(x-x_1)}e^{iq(x_1-x_1')}e^{ip(x_1-x_1')}e^{ip_1(x_1'-y)} \\
&= i(2\pi)^{-8}\int d^4k d^4p d^4p_1 d^4q G^0_{a\lambda}(k)U(q)_{\lambda\lambda',\mu\mu'}G^0_{\lambda'\mu}(p)G^0_{\mu'\beta}(p_1)e^{ikx}e^{-ip_1y}\delta^{(4)}(p+q-k)\delta^4(p_1-q-p) \\
&= (2\pi)^{-4}\int d^4k e^{ik(x-y)}[iG^0_{a\lambda}(k)(2\pi)^{-4}]\int d^4p U(k-p)_{\lambda\lambda',\mu\mu'}G^0_{\lambda'\mu}(p)G^0_{\mu'\beta}(k) \\
&\Rightarrow iG_{ab}(k) = -(2\pi)^{-4}G^0_{a\lambda}(k)\int d^4p U(k-p)_{\lambda\lambda',\mu\mu'}G^0_{\lambda'\mu}(p)G^0_{\mu'b}(k) \qquad (1.97)
\end{aligned}
$$

where we used $\delta^4(p) = (2\pi)^{-4}\int d^4x e^{ipx}$.

we can now state the Feynman rules for the n-th-order contribution to $G_{ab}(k,\omega)$:

1. Draw a1l topologically distinct connected diagtams with n interaction lines and 2n+ 1 directed Green'sfunctions;

2. Assign a direction to each interaction line; associate a directed four-momentum with each line and conserve four-momentum at each vertex;

3. Each green's function corresponds to a factor

$$G^0_{ab}(k,\omega) = \delta_{ab}G^0(k,\omega) \qquad (1.98)$$

4. Each interaction corresponds to a factor $U(q)_{\lambda\lambda',\mu\mu'} = V(q)_{\lambda\lambda',\mu\mu'}$, where the matrix indices are associated with the fermion lines;

5. Perform a spin summation along each continuous particle line including the potential at each vertex;

6. Integrate over the n independent internal four-momenta.

7. Affix a factor $i^n(2\pi)^{-4n}(-1)^F$ where F is the number of closed fermion loops;

8. Internal Green's function is interpreted as $e^{i\omega\eta}G_{ab}(k,\omega)$, $\eta\to 0^+$.

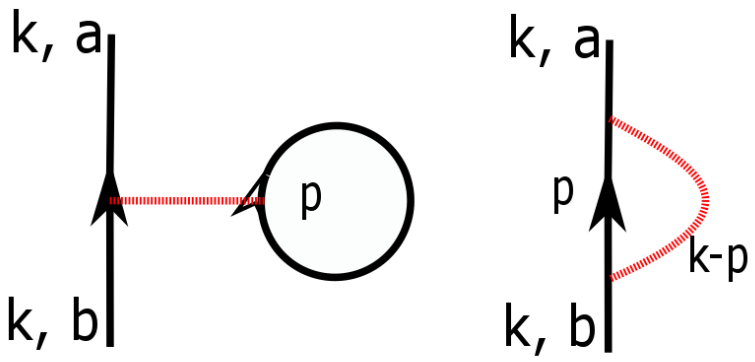


Figure 1.7: Feynman diagrams in the momentum space.

As an example, we calculate diagrams in Fig. 1.3.3:

$$G^1_{ab}(k) = i(-1)(2\pi)^{-4}\int d^4k_1 G^0_{a\lambda}(k)U(0)_{\lambda\lambda',\mu\mu'}G^0_{\lambda' b}(k)G^0_{\mu'\mu}(k_1)e^{i\omega_1\eta}$$
$$+ i(2\pi)^{-4}\int d^4k_1 G^0_{a\lambda}(k)U(k-k_1)_{\lambda\lambda',\mu\mu'}G^0_{\lambda'\mu}(k_1)G^0_{\mu' b}e^{i\omega_1\eta} \quad (1.99)$$
$$= iG^0(k)(2\pi)^{-4}\int d^4k_1[-U(0)_{ab,\mu\mu}G^0(k_1)e^{i\omega_1\eta} + U(k-k_1)_{a\mu,\mu b}G^0(k_1)e^{i\omega_1\eta}]G^0(k) \quad (1.100)$$

1.3.4 Dyson Equation

We have shown how to calculate the Green's function using perturbative expansion. But in many cases, the first few order expansion is not enough, or not accurate enough. So we need some way to evaluate infinite order expansion. Dyson equation is such kind of infinite-order expansion method (Dyson, 1949).

Dyson defines the so-called self-energy function, which actually includes all perturbative corrections to the bare Green's function. Our graphical analysis makes clear that the exact Green's function consists of the unperturbed Green's function plus a11 connected terms with a free Green's function at each end. In formulism, we notice that the full Green's function can be expressed as

$$G_{ab}(x,y) = G^{(0)}_{ab}(x,y) + \int dx_1 dx_2 G^{(0)}_{a\lambda}(x,x_1)\Sigma_{\lambda\mu}(x_1,x_2)G^{(0)}_{\mu b}(x_2,y) \quad (1.101)$$

which defines the self-energy $\Sigma_{\lambda\mu}(x_1,x_2)$. A self-energy insertion is defined as any part of a

Figure 1.8: (Left) Feynman diagram for Dyson equation, where the pink circle stands for the (total) self-energy function. (Right) Feynman diagram for Dyson equation, where the pink circle stands for the proper self-energy function. The thin line is free Green's function, and the thick line is full Green's function.

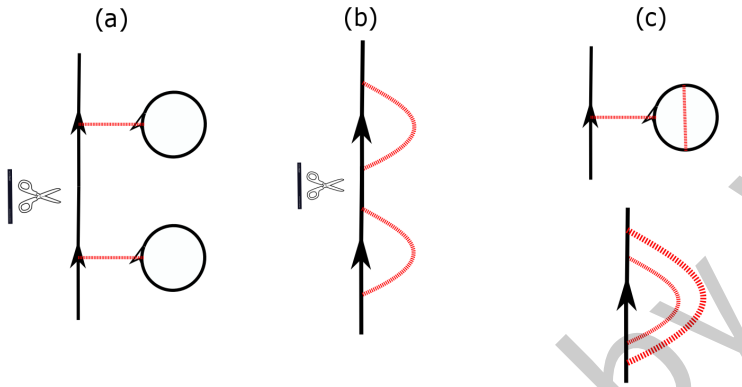


Figure 1.9: Improper (a,b) and proper (c) Feynman diagrams in the real space.

diagram that is connected to the rest of the diagram by two particle lines (one in and one out).

We next introduce the concept of a proper self-energy insertion, which is a self-energy function that cannot be separated into two pieces by cutting a single particle line. For example, Figs. 1.9a,b, all contain improper self-energy insertions, while the remaining terms of Fig.1.9c contain only proper (irreducible) self-energy insertions. By definition, the proper self-energy is the sum of all proper self-energy insertions, and will be denoted $\Sigma^*(x_i, x_j)$. lt follows from these definitions that the self-energy contains of a sum of all possible repetitions of the proper self-energy:

$$\Sigma(x_1, x_2) = \Sigma^*(x_1, x_2) + \int dy_1 dy_2 \Sigma^*(x_1, y_1) G^{(0)}(y_1, y_2) \Sigma^*(y_2, x_2) + \int dy_1 dy_2 \int dz_1 dz_2 \Sigma^*(x_1, y_1) G^{(0)}(y_1, y_2) \Sigma^*(y_2, z_1) G^{(0)}(z_1, z_2) \Sigma^*(z_2, x_2) + \ldots \quad (1.102)$$

as shown in Fig. 1.10.

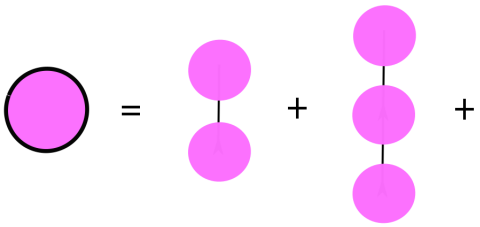


Figure 1.10: Total self-energy (red circle with black line) expressed by the proper self-energy function (red circle without black boundary).

If we inserting the self-energy into the Green's function, we get

$$\begin{aligned} G(x,y) &= G^{(0)}(x,y) + \int dx_1 dx_2 G^{(0)}(x,x_1)\Sigma^*(x_1,x_2)G^{(0)}(x_2,y) + \\ &\int dx_1 dx_2 \int dy_1 dy_2 G^{(0)}(x,x_1)\Sigma^*(x_1,y_1)G^{(0)}(y_1,y_2)\Sigma^*(y_2,x_2)G^{(0)}(x_2,y) + ... \\ \Rightarrow G_{ab}(x,y) &= G^{(0)}_{ab}(x,y) + \int dx_1 dx_2 G^{(0)}_{a\lambda}(x,x_1)\Sigma^*_{\lambda\mu}(x_1,x_2)G_{\mu b}(x_2,y) \end{aligned} \tag{1.103}$$

This is the Dyson equation, which represents the full Green's function in a self-consistent equation. The diagram graph is shown in Fig. 1.8.

Here we see, self-energy is a summation of an infinite number of distinct diagrams, and Dyson equation achieves a great simplification of expression of infinite-order expansions. To evaluation of Green's function is transferred to the evaluation of self-energy function. In practice, one will see people usually target self-energy function directly. Also, in practice, we usually sum subsets of diagrams in the series, instead of all of them. (Actually, to get all diagrams is impossible.) To select different diagrams in the self-energy is usually empirical, and we will show some examples in the following chapters.

Dyson equation doesnot depend on specific representation. If we change it to the momentum space, we get

$$\begin{aligned} \Sigma^*(x,y) &= \int \frac{d^dk}{(2\pi)^d} e^{i\mathbf{k}\cdot(x-y)}\Sigma^*(k) \\ G^{(0)}(x,y) &= \int \frac{d^dk}{(2\pi)^d} e^{i\mathbf{k}\cdot(x-y)}G^{(0)}(k) \\ G(x,y) &= \int \frac{d^dk}{(2\pi)^d} e^{i\mathbf{k}\cdot(x-y)}G(k) \\ \Rightarrow G_{ab}(k) &= G^{(0)}_{ab}(k) + G^{(0)}_{a\lambda}\Sigma^*_{\lambda\mu}(k)G_{\mu b}(k) \\ \Rightarrow G(k) &= \frac{1}{[G^{(0)}]^{-1} - \Sigma^*(k)} \end{aligned} \tag{1.104}$$

At last, we omit the index symbols and write the Dyson equation in a matrix form.

For free electron Green's function $G^{(0)}(k) = 1/(\omega \pm i\eta - \epsilon_k^0)$, we have

$$G(k) = \frac{1}{\omega \pm i\eta - \epsilon_k^0 - \Sigma^*(k)} = \frac{1}{\omega - \epsilon_k - i\gamma_k} \tag{1.105}$$

where $\epsilon_k = \epsilon_k^0 + Re\Sigma(k,\omega), \gamma_k = Im\Sigma$. Here, we see, in the momentum space the full Green's function can be expressed in a simple compact form, which is easy to use. So in many discussion, we would like to change to the momentum space. Physically, the above form of Dyson equation implies that, the full Green's function takes a Lorentz function, akin to that of non-interaction Green's function. This is meaningful for the "quasi-particle" explanation.

1.3.5 Self-energy functions

Dyson equation actually gives a definition of self-energy function. The self-energy function itself contains all perturbative corrections. Once the self-energy function is solved, the full Green's function can be obtained. When calculating the self-energy correction perturbatively, the summation in diagrams is infinite. In practice, we need to select some of important diagrams to calculate. This partial summation method is the focus of this section.

Let us analyze the diagrams for the self-energy function. According to the Dyson equation, we can write the full green's function as $G = 1/(G_0^{-1} - \Sigma)$. Dyson equation only states the solution formally, but it doesnot tell you how to solve or calculate the self-energy. Actually, to solve self-energy is a very tough task. The self-energy by definition includes all possible diagrams, as shown in Fig. **??**. To solve all of Fig. **??** is impossible (sometimes it works only for special models), we have to make some simplifications.

Before going into the detailed discussion, as an example, we recall the first-order correction

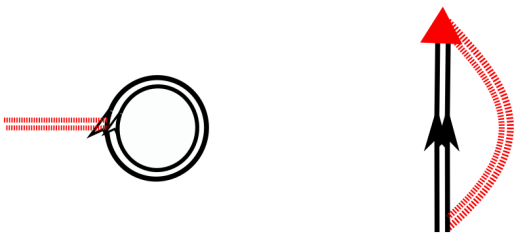


Figure 1.11: Reduced self-energy diagrams.

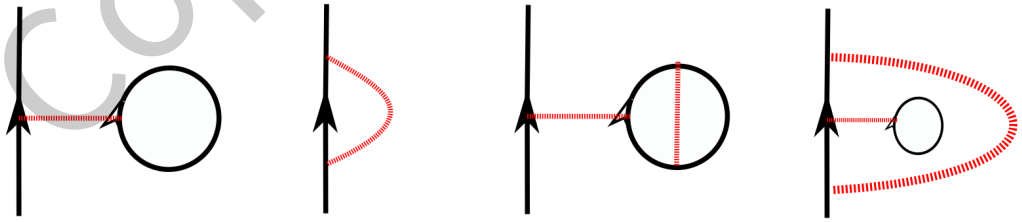
in Green's function under two-particle interaction as

$$\begin{aligned} G^{(1)}_{ab}(x,y) &= i\int d^dx_1 d^dx_2 \{(-1)G^{(0)}_{a\lambda}(x,x_1)V_{\lambda\lambda',\mu\mu'}(x_1,x_2)G^{(0)}_{\lambda' b}(x_1,y)G^{(0)}_{\mu\mu'}(x_2,x_2)+ \\ &\quad G^{(0)}_{a\lambda}(x,x_1)V_{\lambda\lambda',\mu\mu'}(x_1,x_2)G^{(0)}_{\lambda'\mu}(x_1,x_2)G^{(0)}_{\lambda' b}(x_2,y)\} \\ &= \int d^dx_1 d^dx_2 G^{(0)}_{a\lambda}(x,x_1)[\Sigma^{(1)}_{\lambda\lambda'}(x_1,x_2)]G^{(0)}_{\lambda' b}(x_2,y) \\ \Sigma^{(1)}_{\lambda\lambda'}(x_1,x_2) &\equiv -i\delta(x_1-x_2)\int d^d\xi V_{\lambda\lambda'\mu\mu'}(x_1-\xi)G^{(0)}_{\mu\mu'}(\xi,\xi) + iV_{\lambda\lambda'\mu\mu'}(x_1-x_2)G^{(0)}_{\mu\mu'}(x_1,x_2)] \end{aligned}$$

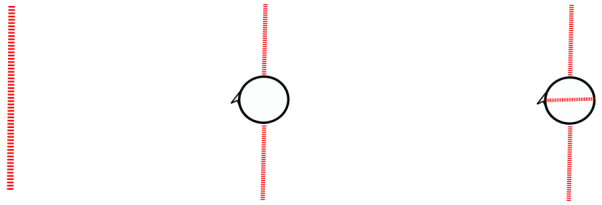
If we stop here, that means we only keep the first-order correction in self-energy function, but neglect all others.

In principle, we can classify all self-energy corrections into two groups, as shown in Fig. 1.11. And this simplified diagrams are made of three building blocks: 1) particle line; 2) interaction line; 3) vertex (representing two particle lines and one interaction line), as shown below:

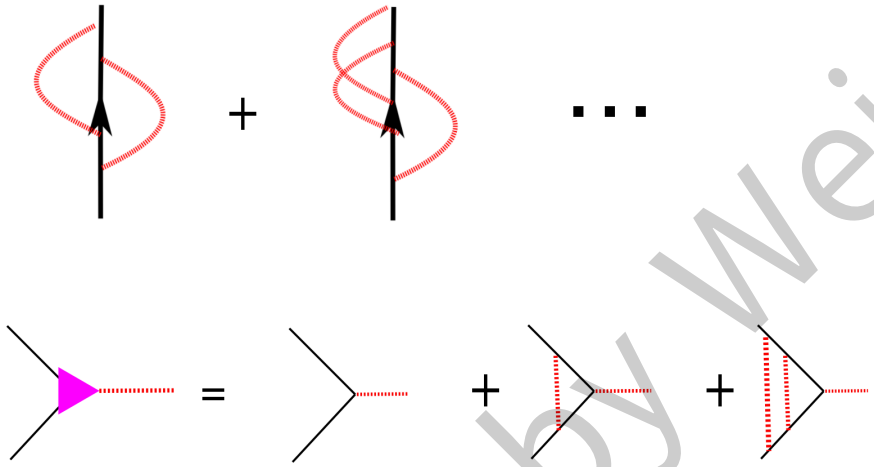
- Particle line with all possible corrections (so that the result is replacing single single G_0 with double line G):



- Interaction line with all possible corrections (inserting bubble diagrams into the interaction line):



- vertex line with all "corner" corrections (those cannot separate into particle line or interaction line):



By combining these corrected diagrams (double lines means the corrected particle line and interaction line), we conclude that all possible diagrams (for two-particle interactions) can be represented by Fig. 1.11.

As we elucidated above, in principal all diagrams have been included in Fig. 1.11. But it is not realistic to calculate all diagrams explicitly. Practically, we have to select some of diagrams to calculate (according to the physical picture or condition). In the following subsections, we introduce some approximated methods that are widely used in literatures: Hartee-Fock, RPA. In the first method, a small set of proper self-energy insertions is reinterpreted, so that the particle lines represent exact Green's functions G instead of noninteracting Green's functions G_0. These approximations are therefore self-consistent. In contrast, the second approach retains a selected (infinite) class of proper self-energy insertions, expressed in terms of G_0.